MATH 285: Stochastic Processes

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Today: Irreducible Markov chains. Random walks on graphs

- Homework 1 is due on Friday, January 14, 11:59 PM

Classification of states : recurrence and transience

Let (X_n) be a Markov chain with state space S.

Def 4.1 A state $i \in S$ is called recurrent if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1$$

A state $i \in S$ is called transient if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 0$$

Denote $T_i := T_{i,2} = \min\{n > 0 : X_n = i\}$ and $r_i := \mathbb{P}_i[T_i < \infty]$

Theorem 4.2

Let $i \in S$. Then

(1) i is recurrent $\Leftrightarrow r_i = 1 \Leftrightarrow \sum_{n=0}^{\infty} p_n(i,i) = \infty$

(2) i is transient $\Leftrightarrow r_i < 1 \Leftrightarrow \sum_{n=0}^{\infty} p_n(i,i) < \infty$.

Recurrence and transience of RW

Example 4.5

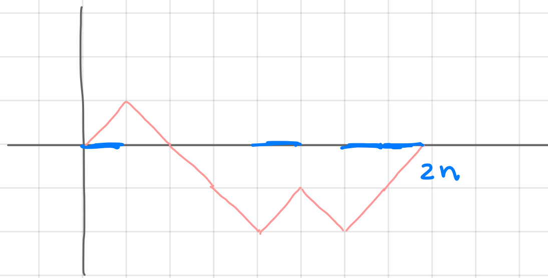
Let (X_n) be a random walk on $\mathbb{Z}$, $p(i,j)=\begin{cases} p, & j=i+1 \\ 1-p, & j=i-1 \\ 0, & \text{otherwise} \end{cases}$

Fix $i \in \mathbb{Z}$. Is i recurrent or transient?

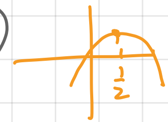
Use the $\sum_{n=0}^{\infty} p_n(i,i)$ criterion.

Notice that $p_n(i,i)=0$ if n is odd

Goal: compute $\sum_{n=0}^{\infty} p_{2n}(i,i)$



$$p_{2n}(i,i) = \binom{2n}{n} p^n (1-p)^n$$ (trivial for $p=0$ or $p=1$)



Case 1: $p \in (0,1)$, $p \neq \frac{1}{2}$. Then $p(1-p) < \frac{1}{4}$

$$\sum_{n=0}^{\infty} p_{2n}(i,i) = \sum_{n=0}^{\infty} \binom{2n}{n} (p(1-p))^n < \sum_{n=0}^{\infty} 4^n (p(1-p))^n < \infty$$

$$\binom{2n}{n} < 4^n = \sum_{k=0}^{2n} \binom{2n}{k}$$ $\Rightarrow$ all states are transient

Recurrence and transience of RW

Case 2: $p = \frac{1}{2}$

$$\binom{2n}{n} = \frac{(2n)!}{n!\, n!} \leftarrow \text{use Stierling's approximation}$$

$$n! \sim \sqrt{2\pi n}\, \frac{n^n}{e^n}$$

$$\binom{2n}{n} \sim \sqrt{\cancel{4}\pi n}\, \frac{(2\cancel{n})^{2n}}{\cancel{e^{2n}}} \cdot \frac{\cancel{e^{2n}}}{\cancel{2}\pi n\ \cancel{n^{2n}}} = 2^{2n} \cdot \frac{1}{\sqrt{\pi n}}$$

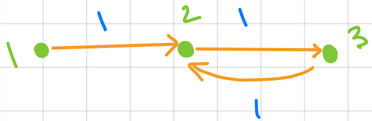
$$\sum_{n=0}^{\infty} p_n(i,i) \sim \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}}\, \cancel{2^{2n}} \cdot \cancel{\left(\frac{1}{4}\right)^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = +\infty$$

$\Rightarrow$ all states are recurrent

Irreducibility

Is it always true that either all states are recurrent or all states are transient? NO

Example



1 is transient

2, 3 are recurrent

Def 4.7 Markov chain is called irreducible if for any $i, j \in S$ there exists $n \in \mathbb{N}$ s.t. $p_n(i,j) > 0$

Prop. 4.8 If (X_n) is irreducible, then either all states are recurrent or all states are transient.

Proof. Suppose i is transient, $j \in S$, $p_{n_0}(i,j) > 0$, $p_{n_1}(j,i) > 0$

Then $\forall\, m \in \mathbb{N}$ $\quad p_{n_0+m+n_1}(i,i) \geq p_{n_0}(i,j)\, p_m(j,j)\, p_{n_1}(j,i)$

$$\sum_{m=0}^{\infty} p_m(j,j) \leq \sum_{m=0}^{\infty} \frac{1}{p_{n_0}(i,j)} \cdot \frac{1}{p_{n_1}(j,i)}\, p_{n_0+m+n_1}(i,i) < \infty \Rightarrow j \text{ is transient}$$

■

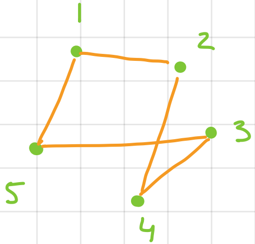
Graphs

Def 5.1 A graph $G=(V,E)$ is a collection of vertices V and relations E on $V\times V$ (which we call edges).

For $x,y\in V$ we write $x\sim y$ to mean $(x,y)\in E$.

E is assumed to be anti-reflexive ($x\not\sim x$, no loops) and symmetric (if $x\sim y$ then $y\sim x$, indirected graph).

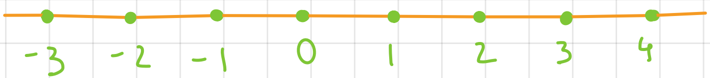
Example



$V=\{1,2,3,4,5\}$

$E=\{(1,2),(2,1),(2,4),(4,2),(4,3),(3,4)$
$(3,5),(5,3),(5,1),(1,5)\}$

Example



$V=\mathbb{Z}$

$E=\{(i,i+1),(i,i-1):i\in\mathbb{Z}\}$

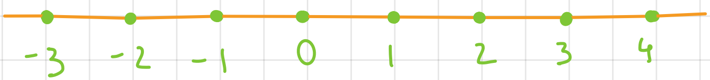
Valence of a vertex $x\in V$: $v_x=\#\{y\in V: x\sim y\}$

Simple random walks of graphs

Def. 5.2 The simple random walk on the graph $G=(V,E)$ is the Markov chain (X_n) with state space V and transition probabilities $p(i,j)$ s.t. $p(i,j)>0$ $i\sim j$ and $p(i,j)=0$ $i\not\sim j$. (X_n) is called symmetric if $p(i,j)=\frac{1}{v_i}$ for all j s.t. $i\sim j$.

Example 5.3 RW on $\mathbb{Z}$

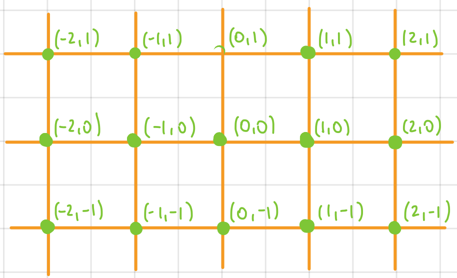
$$p(i,j)=\begin{cases}\frac{1}{2}, & j=i+1\\ \frac{1}{2}, & j=i-1\\ 0, & \text{otherwise}\end{cases}$$



Example 5.4.

RW on $\mathbb{Z}^d$

$$\|x\|_1=\sum_{m=1}^{d}|x_m|$$



$$V=\mathbb{Z}^d=\{\underbrace{(i_1,\dots,i_d)}_{\bar{i}}: i_m\in\mathbb{Z}\}$$

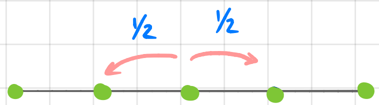
$\bar{i}\sim\bar{j}$ iff $\|\bar{i}-\bar{j}\|_1=1$

$$\Rightarrow\quad v_i=\frac{1}{2d}$$

SRW on $\mathbb{Z}^d$

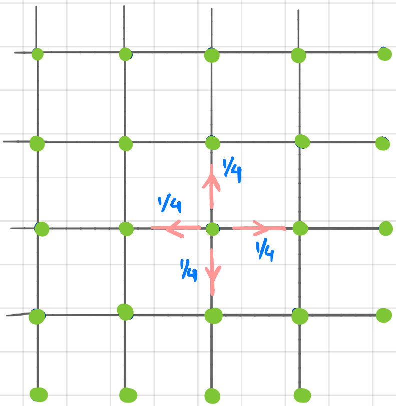
Remark For any $d \in \mathbb{N}$, simple random walks on $\mathbb{Z}^d$ are irreducible $\Rightarrow$ all states are in the same class

SSRW on $\mathbb{Z}^d$, $d \in \{1,2,3\}$



~~transient~~

recurrent



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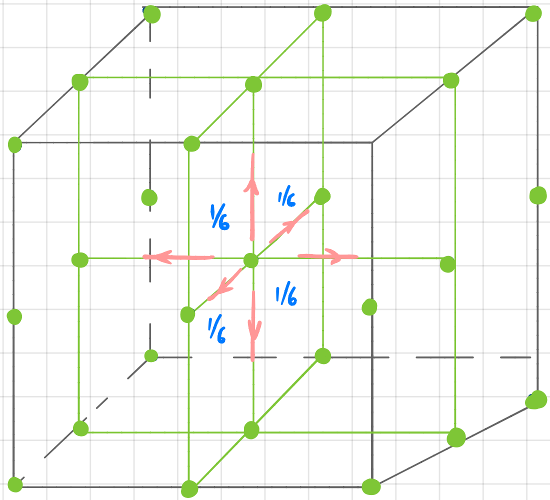


transient

recurrent

Simple symmetric RW on $\mathbb{Z}^3$

As for $d=1$, $p_n(i,i)=0$ if n is odd

Goal: determine if $\sum_{n=0}^{\infty} p_{2n}(i,i)$ is finite or not.

Take $i=\bar{0}=(0,0,0)$ for simplicity.

$$p_{2n}(\bar{0},\bar{0}) = \#\{\text{paths from } \bar{0} \text{ to } \bar{0} \text{ in } 2n \text{ steps}\} \cdot \left(\frac{1}{6}\right)^{2n}$$

i steps $(+1,0,0)$

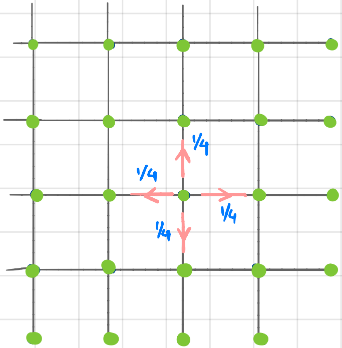
i steps $(-1,0,0)$

j steps $(0,+1,0)$

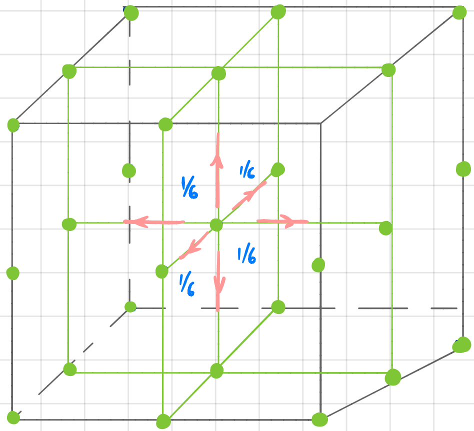
j steps $(0,-1,0)$

k steps $(0,0,+1)$

k steps $(0,0,-1)$



$$2i+2j+2k=2n$$



Simple symmetric RW on $\mathbb{Z}^3$

Step 1: $\#\{\text{paths from } \bar{0} \text{ to } \bar{0} \text{ of length } 2n\} = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{2n}{i,i,j,j,k,k}$

$$P_{2n}(\bar{0},\bar{0}) = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \frac{(2n)!}{(i!\,j!\,k!)^2} \cdot \left(\frac{1}{6}\right)^{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \left(\frac{n!}{i!\,j!\,k!}\right)^2 \cdot \left(\frac{1}{3}\right)^{2n}$$

Step 2: $\displaystyle\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k} = 3^n$, so $\displaystyle\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^n = 1$

$\forall i \quad a_i^2 \leq a_i M$

Step 3: If $a_i \geq 0$ and $a_i \leq M$, then $\sum a_i^2 \leq M \sum a_i$

and thus $\displaystyle\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k}^2 \left(\frac{1}{3}\right)^{2n} \leq \max_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^n \cdot 1$

Simple symmetric RW on $\mathbb{Z}^3$

Steps 1-3 imply that

$$p_{2n}(\bar{0},\bar{0}) \le \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \max_{\substack{i,j,k\ge 0\\ i+j+k=n}} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^{n} \quad (*)$$

Step 4: If $n = 3m$, then $\displaystyle\max_{\substack{i,j,k\ge 0\\ i+j+k=n}} \binom{n}{i,j,k} = \binom{n}{m,m,m}$

Step 5:

$$\frac{(3m)!}{m!\,m!\,m!}\left(\frac{1}{3}\right)^{3m} \sim \frac{\cancel{(3m)^{3m}}}{\cancel{e^{3m}}} \cdot \frac{\cancel{e^{3m}}}{\cancel{m^{3m}}} \cancel{\left(\frac{1}{3}\right)^{3m}} \frac{\sqrt{2\pi n}}{(2\pi m)^{3/2}}$$

$$= \frac{\sqrt{2\pi n}}{(2\pi m)^{3/2}}$$

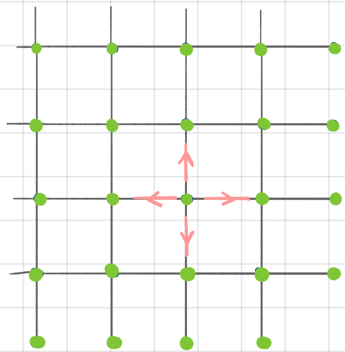
Steps 4-5 + $(*)$ + asymptotics for $\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}}$ gives

$$p_{6m}(\bar{0},\bar{0}) \sim \frac{1}{\sqrt{\pi n}} \cdot \frac{\sqrt{2\pi n}}{(2\pi m)^{3/2}} = \frac{1}{2(\pi m)^{3/2}} \text{ and } \sum_{m=0}^{\infty} p_{6m}(\bar{0},\bar{0}) < \infty$$

Simple symmetric RW on $\mathbb{Z}^3$

Step 6: $p_{6m}(\bar{0},\bar{0}) \geq \left(\frac{1}{6}\right)^2 p_{6m-2}(\bar{0},\bar{0}) \quad \forall\, m \in \mathbb{N}$

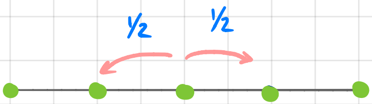
$p_{6m}(\bar{0},\bar{0}) \geq \left(\frac{1}{6}\right)^4 p_{6m-4}(\bar{0},\bar{0}) \quad \forall\, m \in \mathbb{N}$



Conclusion: $\sum_{n=0}^{\infty} p_{2n}(\bar{0},\bar{0}) \leq \left(1+6^2+6^4\right) \sum_{m=0}^{\infty} p_{6m}(\bar{0},\bar{0}) < \infty$

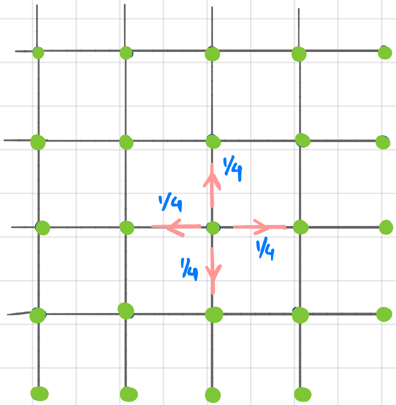
All states of a SSRW on $\mathbb{Z}^3$ are transient

SSRW on $\mathbb{Z}^d$, $d \in \{1, 2, 3\}$



~~transient~~

recurrent



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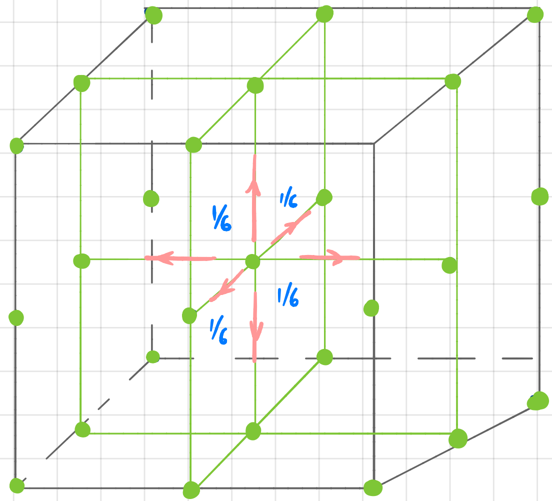

transient

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