MATH 285: Stochastic Processes

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Today: Hitting times. First step analysis. Stopping times

- Homework 1 is due on Friday, January 14, 11:59 PM

Expected hitting times

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities $p(i,j)$ and state space S.

Notation: $\mathbb{P}_i[Y] = \mathbb{P}[Y \mid X_0 = i]$, $\mathbb{E}_i[Y] = \mathbb{E}[Y \mid X_0 = i]$

Let $A \subset S$, $\tau_A := \min\{n \geq 0 : X_n \in A\}$

Q: How long (on average) does it take to reach A?

Compute $\mathbb{E}_i[\tau_A] =$

By definition, $\mathbb{E}_i[Y] = \sum_{k=1}^{\infty} k\,\mathbb{P}[Y = k \mid X_0 = i]$ $\quad (Y \in \{0,1,2,\dots\})$

First step analysis (conditioning on the first step)

$$g(i) = \mathbb{E}_i[\tau_A] =$$

Expected hitting times

If $i \in A$, then $g(i) = 0$. Suppose $i \notin A$.

Then

$$\mathbb{P}[\tau_A = k \mid X_1 = j, X_0 = i] = \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-1} \notin A, X_k \in A \mid X_1 = j, X_0 = i]$$

$$= \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-2} \notin A, X_{k-1} \in A \mid X_0 = j]$$

$$= \mathbb{P}[\tau_A = k-1 \mid X_0 = j]$$

Compute the expectation

$$g(i) = \sum_{j \in S} \mathbb{E}[\tau_A \mid X_1 = j, X_0 = i]\, \mathbb{P}[X_1 = j \mid X_0 = i]$$

$$=$$

$$=$$

Expected hitting times

Conclusion:

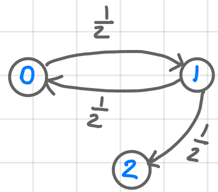
$$\begin{cases} g(i) = 1 + \sum_{j\in S} p(i,j)\, g(j) & \text{if } i \notin A \\ g(i) = 0 & \text{if } i \in A \end{cases}$$

Example 3.2 On average how many times do we need to toss a coin to get two consecutive heads?

Denote by the number of consecutive heads after n^{th} toss.

$X_n \in \{0, 1, 2\}$,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



$g(2) = 0$ $\quad g(1) =$ $\qquad\qquad g(0) =$

$g(0) =$ $\qquad\qquad g(1) =$ $\quad g(0) =$

Starting from state 0 it takes on average 6 tosses to reach state 2.

Stopping Times

Def 3.3 Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process.

A stopping time is a

such that for each n the event $\{T=n\}$ depends

only on

Examples $T_1 = \min\{n\geq 0 : X_n = i\}$ is a stopping time

$\{T_1 = n\} =$

$T_2 = \max\{n\geq 0 : X_n = i\}$ is not a stopping time

$\{T_2 = n\} =$

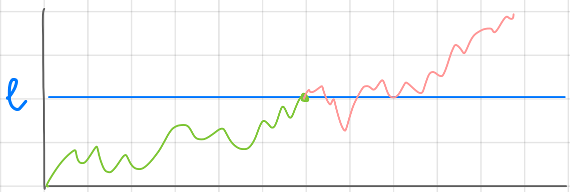
Recall Markov property: If (X_n) is Markov(λ, P), then conditional on $X_m = \ell$, the process $(X_{m+n})_{n\in\mathbb{N}}$ is Markov(δ_ℓ, P) independent of $X_0, X_1, \dots, X_m$

Strong Markov property

Proposition 3.6 Let (X_n) be a time-homogeneous Markov chain with state space S and transition probabilities $p(i,j)$. Let T be a stopping time, $\ell \in S$ and $\mathbb{P}[X_T = \ell] > 0$. Then, conditional on $X_T = \ell$, $(X_{T+n})_{n\geq 0}$ is a time-homogeneous

independent of $X_0, \dots, X_T$. In other words, if A is an event that depends only on $X_0, X_1, \dots, X_T$ and $\mathbb{P}[A \cap \{X_T = \ell\}] > 0$ then for all $n \geq 0$ and all $i_0, i_1, \dots, i_n \in S$

$$\mathbb{P}[X_{T+1} = i_1, X_{T+2} = i_2, \dots, X_{T+n} = i_n \mid A \cap \{X_T = \ell\}] =$$



Proof. Use the partition $\{\{T = m\}\}_{m=0}^{\infty}$ and apply Markov property (see the notes)

Classification of states : recurrence and transience

Let (X_n) be a Markov chain with state space S.

Def 4.1 A state $i \in S$ is called recurrent if

A state $i \in S$ is called transient if

Remark

Let $T_{i,k}$ = time X_n (starting from i) visits state i k^{th} time

$T_{i,1} = 0$, $T_{i,k+1} =$

Then, for $k \geq 2$, $T_{i,k}$ are stopping times. Indeed,

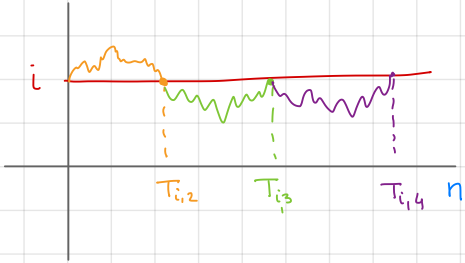
$\{T_{i,2} = m\} =$

$$\{T_{i,k} = m\} = \bigsqcup_{l=k-2}^{m-1} \{T_{i,k-1} = l, T_{i,k} = m\} = \bigsqcup_{l=k-2}^{m-1} \{\underbrace{T_{i,k-1} = l}_{\text{depends on } X_0, \dots, X_l}, X_{l+1} \neq i, \dots, X_{m-1} \neq i, X_m = i\}$$

Classification of states : recurrence and transience

Denote $T_i := T_{i,2} =$ and

$r_i :=$



Theorem 4.2

Let $i \in S$. Then

(1) i is recurrent $\Leftrightarrow$ $\Leftrightarrow$

(2) i is transient $\Leftrightarrow$ $\Leftrightarrow$

Proof. Step 1: By the Strong Markov property

$$\mathbb{P}_i[T_{i,k+1} < \infty \mid T_{i,k} < \infty] =$$

$$\mathbb{P}_i[T_{i,k+1} < \infty] =$$

Step 2: Denote $N_i :=$ $\leftarrow$ # times (X_n) visits state i

$\forall\ k \geq 1$, $\{N_i \geq k\} =$, so $\mathbb{P}_i[N_i \geq k] = \mathbb{P}_i[T_{i,k} < \infty] =$

Classification of states : recurrence and transience

Thus $\mathbb{E}_i[N_i] = \sum_{k=1}^{\infty} \mathbb{P}_i[N_i \geq k] =$

$$\mathbb{E}_i[N_i] = \mathbb{E}_i\left[\sum_{n=0}^{\infty} \mathbb{1}_{X_n=i}\right] = \sum_{n=0}^{\infty} \mathbb{P}_i[X_n = i] =$$

Since $r_i \in [0,1]$, $\sum_{\ell=0}^{\infty} r_i^\ell = \infty \iff$, $\sum_{\ell=0}^{\infty} r_i^\ell < \infty \iff$

Step 3: $r_i = 1 \iff \forall k \;\; \mathbb{P}_i[N_i \geq k] = 1$, i.e., i is

Step 4: $r_i < 1 \iff \mathbb{P}_i[N_i \geq k] = r_i^k \to 0$, $k \to \infty$,

so $\mathbb{P}_i[N_i = \infty] = 0$, i.e., i is

$$\sum_{n=0}^{\infty} p_n(i,i) = \sum_{\ell=0}^{\infty} r_i^\ell =$$

$\blacksquare$

Recurrence and transience of RW

Example 4.5

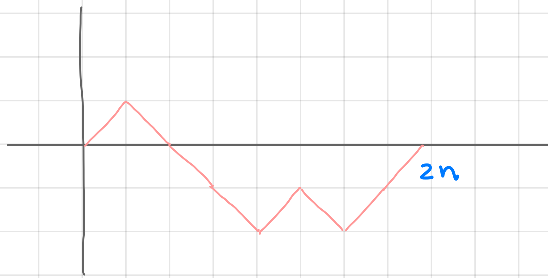
Let (X_n) be a random walk on $\mathbb{Z}$, $p(i,j) = \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$

Fix $i \in \mathbb{Z}$. Is i recurrent or transient?

Use the $\sum_{n=0}^{\infty} p_n(i,i)$ criterion.

Notice that $p_n(i,i) = 0$ if n is odd

Goal: compute $\sum_{n=0}^{\infty} p_{2n}(i,i)$



$p_{2n}(i,i) =$ (trivial for $p=0$ or $p=1$)

Case 1: $p \in (0,1)$, $p \neq \frac{1}{2}$. Then $p(1-p) < \frac{1}{4}$

$$\sum_{n=0}^{\infty} p_{2n}(i,i) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(p(1-p)\right)^n$$

$\binom{2n}{n} < 4^n$ $\Rightarrow$ all states are

Recurrence and transience of RW

Case 2: $p = \frac{1}{2}$

$$\binom{2n}{n} = \frac{(2n)!}{n!\, n!} \leftarrow \text{use Stierling's approximation}$$

$$n! \sim \sqrt{2\pi n}\, \frac{n^n}{e^n}$$

$$\binom{2n}{n} \sim$$

$$\sum_{n=0}^{\infty} p_n(i,i) =$$

$\Rightarrow$ all states are