MATH 285: Stochastic Processes

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Today: Hitting times. First step analysis

- Test Homework on Gradescope

Initial distribution and transition matrix

Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with finite state space $S = \{s_1, s_2, \dots, s_{|S|}\}$ $(= \{1, 2, 3, \dots, |S|\})$

Distribution of X_n is a vector $(\mathbb{P}[X_n = 1], \mathbb{P}[X_n = 2], \dots, \mathbb{P}[X_n = |S|])$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{|S|})$ be the distribution of X_0, i.e., $\mathbb{P}[X_0 = i] = \lambda_i$. Let P be the transition matrix of (X_n).

Q: What is the distribution of X_n?

X_1 : $\mathbb{P}[X_1 = j] =$

Distribution of X_1 is given by

X_n : $\mathbb{P}[X_n = j] = \sum_{i=1}^{|S|} \mathbb{P}[X_n = j \mid X_0 = i]\, \mathbb{P}[X_0 = i] =$

Distribution of X_n is given by

We will say that (X_n) is Markov(λ, P)

Markov property "future is independent of the past"

Prop 2.5 Let (X_n) be a time-homogeneous MC with discrete state space S and transition probabilities $p(i,j)$. Fix $m \in \mathbb{N}$, $\ell \in S$, and suppose that $\mathbb{P}[X_m = \ell] > 0$. Then conditional on $X_m = \ell$, the process $(X_{m+n})_{n \in \mathbb{N}}$ is Markov with transition probabilities $p(i,j)$, initial distribution

$$\begin{array}{cccccccccc} 1 & & & \ell-1 & \ell & \ell+1 & & & \\ (0, & \dots, & & 0, & 1, & 0, & \dots, & 0) \end{array}$$

and independent of the random variables $X_0, \dots, X_m$, i.e. if A is an event determined by $X_0, X_1, \dots, X_m$ and $\mathbb{P}[A \cap \{X_m = \ell\}] > 0$ then for all $n \geq 0$

$$\mathbb{P}[X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n} \mid A \cap \{X_m = \ell\}] =$$

Proof. Enough to show that

$$\mathbb{P}[\{X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}, X_m = \ell\} \cap A] = p(\ell, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n}) \, \mathbb{P}[A \cap \{X_m = \ell\}]$$

Markov property

- Let $A = \{X_0 = i_0, \dots, X_m = \ell\}$. Then

$\mathbb{P}[X_0 = i_0, \dots, X_m = \ell, X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}\} =$

,

$\mathbb{P}[X_0 = i_0, \dots, X_m = \ell] = \mathbb{P}[X_0 = i_0]\, p(i_0, i_1) \cdots p(i_{m-1}, \ell)$

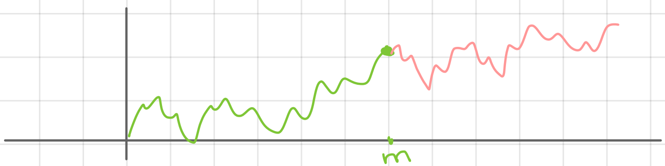
- Any set A determined by $X_0, \dots, X_m$ is a disjoint union of the events of the form $\{X_0 = i_0, \dots, X_m = i_m\}$.

E.g. $\mathbb{P}[\{X_{m+1} = i_{m+1}, \dots, X_{m+n} = i_{m+n}\} \cap (A_1 \sqcup A_2) \cap \{X_m = \ell\}]$

$=$

$=$

So $(*)$ holds for any event A.



Hitting times

Q1: When is the first time the process enters a certain set?

For $A \subset S$, compute

Q2: For $A, B \subset S$, $A \cap B = \emptyset$ find the probability

Start with Q2

- trivial:

- take $i \notin A \cup B$; "first step analysis":

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i] =$$

By the Markov property

$$\mathbb{P}[\tau_A < \tau_B \mid X_0 = i, X_1 = j] =$$

Hitting times

We conclude that

$$h(i) = \qquad\qquad\qquad\qquad (**)$$

This gives a system of linear equations + boundary conditions

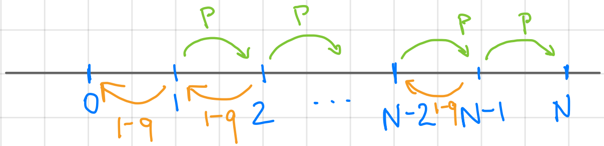
$$h(i) = \begin{cases} 1, & i \in A \\ 0, & i \in B \end{cases} \qquad\qquad (***)$$

If S is finite, denote $\bar{h} := (h(1), h(2), \ldots, h(|S|))$. Then $(**)$ becomes

Example 2.6 (X_n) random walk on $\{0, 1, 2, \ldots, N\}$, not necessarily symmetric, $p(i, i+1) = q$, $p(i, i-1) = 1-q$, $q \in [0,1]$

Let $i \in \{1, 2, \ldots, N-1\}$. Compute

$\mathbb{P}[X_n \text{ reaches } N \text{ before } 0 \mid X_0 = i]$



Hitting times for random walks

Denote $A=\{N\}$, $B=\{0\}$. Need $\mathbb{P}[\tau_A<\tau_B \mid X_0=i]=h(i)$

- boundary conditions

Consider $0<i<N$

- recall $p(i,j)=\begin{cases} q, & j=i+1 \\ 1-q, & j=i-0 \\ 0, & \text{otherwise}\end{cases}$, so $(**)$ becomes

$$h(i)=\sum_{j\in S} p(i,j)\,h(j)$$

$h(i)=$

$\forall i\in\{1,\dots,N-1\}$

- if $q=0$, then $h(i)=$
- if $q=1$, then $h(i)=$
- if $q\in(0,1)$, denote $\Delta h(i):=h(i)-h(i-1)$, $\theta:=\dfrac{1-q}{q}$

Hitting times for random walks

$$\oplus \begin{cases} \Delta h(2) = \\ \Delta h(3) = \\ \vdots \\ \Delta h(N) = \end{cases}$$

Take the sum of the first i equations

LHS : $\Delta h(1) + \Delta h(2) + \cdots + \Delta h(i) =$

RHS :

$\Rightarrow \quad \forall \ i \in \{2, 3, \ldots, N\} \qquad h(i) =$

$h(N) = 1 =$ $\qquad \Rightarrow \ \Delta h(1) =$

$\Rightarrow \ h(i) =$ $\qquad$ for $i \in \{1, \ldots, N-1\}$

Gambler's ruin

Suppose you have 100 \$, at each game you bet 1 \$, and you stop either when your fortune reaches 200 \$ or when you lose everything. [$N = 200$, $h(100) - ?$]

(fair game) If probability of winning is 0.5 ($q = 0.5$)

then $\theta = \frac{0.5}{0.5} = 1$, $h(100) = \frac{100}{200} = \frac{1}{2} = 0.5$

(real gambling) If probability of winning is $\frac{18}{38}$ ($q = 0.474$)

then $h(100) = \frac{1 - \theta^{100}}{1 - \theta^{200}} =$

Expected hitting times

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities $p(i,j)$ and state space S.

Notation: $\mathbb{P}_i[Y] = \mathbb{P}[Y \mid X_0 = i]$, $\mathbb{E}_i[Y] = \mathbb{E}[Y \mid X_0 = i]$

Let $A \subset S$, $\tau_A := \min\{n \geq 0 : X_n \in A\}$

Q1: How long (on average) does it take to reach A?

Compute $\mathbb{E}_i[\tau_A] =$

By definition, $\mathbb{E}_i[Y] = \sum_{k=1}^{\infty} k\, \mathbb{P}[Y = k \mid X_0 = i]$ $\quad (Y \in \{0, 1, 2, \dots\})$

First step analysis (conditioning on the first step)

$$g(i) = \mathbb{E}_i[\tau_A] =$$

Expected hitting times

If $i \in A$, then $g(i) = 0$. Suppose $i \notin A$.

Then

$$\mathbb{P}[\tau_A = k \mid X_1 = j, X_0 = i] = \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-1} \notin A, X_k \in A \mid X_1 = j, X_0 = i]$$

$$= \mathbb{P}[X_0 \notin A, X_1 \notin A, \dots, X_{k-2} \notin A, X_{k-1} \in A \mid X_0 = j]$$

$$= \mathbb{P}[\tau_A = k-1 \mid X_0 = j]$$

Compute the expectation

$$g(i) = \sum_{j \in S} \mathbb{E}[\tau_A \mid X_1 = j, X_0 = i]\, \mathbb{P}[X_1 = j \mid X_0 = i]$$

$$=$$

$$=$$

Expected hitting times

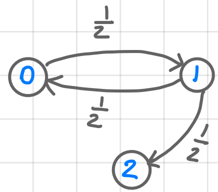
Conclusion:

$$\begin{cases} g(i) = 1 + \sum_{j \in S} p(i,j)\, g(j) & \text{if } i \notin A \\ g(i) = 0 & \text{if } i \in A \end{cases}$$

Example 3.2 On average how many times do we need to toss a coin to get two consecutive heads?

Denote by X_n the number of consecutive heads after n^{th} toss.

$X_n \in \{0, 1, 2\}$,
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



$g(2) =$

$g(1) =$

$g(0) =$

$g(0) =$

$g(1) =$

$g(0) = 6$

Starting from state 0 it takes on average 6 tosses to reach state 2.