MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Martingale convergence theorem

- Homework 7 is due on Friday, March 11, 11:59 PM

The martingale convergence theorem

Theorem 26.1 Let $(X_n)_{n \geq 0}$ be a martingale, and suppose there exists $C \geq 0$ such that $\mathbb{P}[X_n \geq -C] = 1$ for all n.
Then there is a random variable X_∞ such that

$$\mathbb{P}\left[\lim_{n\to\infty} X_n = X_\infty\right] = 1$$

Proof (1) Enough to prove for $C = 0$

Consider $Y_n = X_n + C$. Then (Y_n) is a martingale, $Y_n \geq 0$, and $\lim_{n\to\infty} Y_n = Y_\infty$ if and only if $\lim_{n\to\infty} X_n = Y_\infty - C$

Assume that $X_n \geq 0$

(2) $\mathbb{P}\left[\max_{n\geq 0} X_n < \infty\right] = 1$

- (X_n) is a nonnegative martingale, therefore by

The martingale convergence theorem

- Doob's Maximal inequality for any $N \in \mathbb{N}$

$$\mathbb{P}\left[\max_{0 \le n \le N} X_n \ge a\right] \le \frac{\mathbb{E}[X_N]}{a} = \frac{\mathbb{E}[X_0]}{a}$$

- Take the limit $N \to \infty$ (monotonicity of $\mathbb{P}$)

$$\lim_{N \to \infty} \mathbb{P}\left[\max_{0 \le n \le N} X_n \ge a\right] = \mathbb{P}\left[\max_{n \ge 0} X_n \ge a\right] \le \frac{\mathbb{E}[X_0]}{a}$$

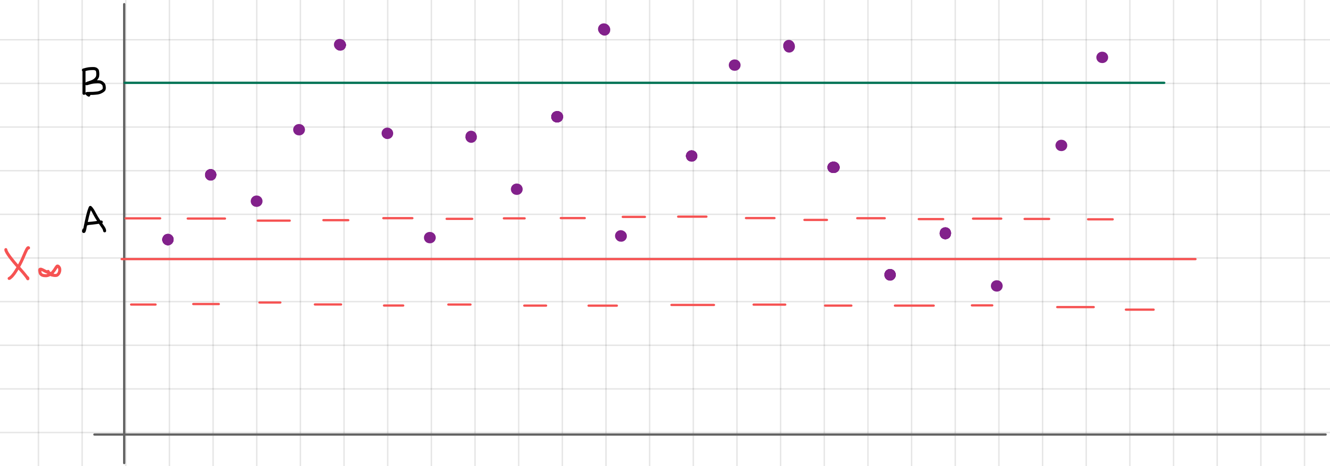
- Take the limit $a \to \infty$

$$\lim_{a \to \infty} \mathbb{P}\left[\max_{n \ge 0} X_n \le a\right] = \mathbb{P}\left[\max_{n \ge 0} X_n < \infty\right] \ge \lim_{a \to \infty}\left(1 - \frac{\mathbb{E}[X_0]}{a}\right) = 1$$

(3) Each trajectory $(X_n(\omega))$ has a convergent subsequence $(X_{n_k}(\omega))$, denote the limit $X_\infty(\omega)$

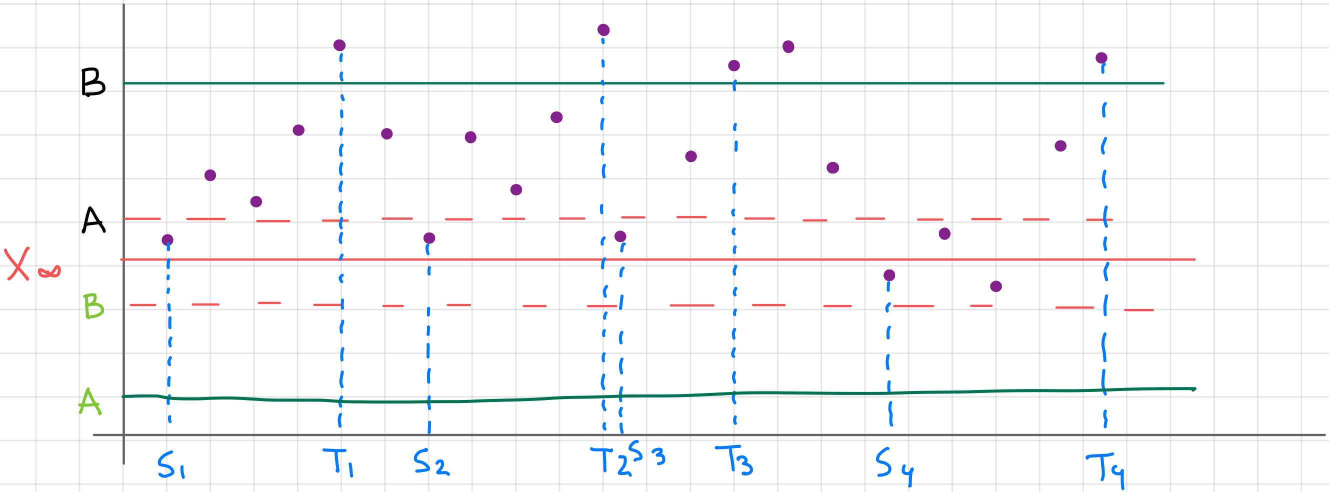
The martingale convergence theorem

(4) If $(X_n(\omega))$ is not convergent, there are infinitely many terms $X_n(\omega)$ away from $X_\infty(\omega)$



If $(X_n(\omega))$ is not convergent, there are $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$ such that there are infinitely many terms $X_n(\omega) \geq B$ and infinitely many terms $X_n(\omega) \leq A$.

The martingale convergence theorem



For any $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$ denote

$S_1 = \min\{k : X_k \leq a\}$, $T_n = \min\{k > S_n : X_k \geq b\}$, $S_n = \min\{k > T_{n-1} : X_k \leq a\}$

(S_n, T_n) denotes an (A, B)-upcrossing

(5) If $(X_n(\omega))$ is not convergent, then there exist infinitely many (A, B) upcrossings for some $A < B$

The martingale convergence theorem

Fix A, B. Denote $U_n := \max\{k : T_k \le n\}$, number of (A,B)-upcrossings before time n.

Denote $U = \lim_{n\to\infty} U_n \in \mathbb{N} \cup \{\infty\}$, total number of (A,B)-upcrossings.

$\{S_k < j\} = \{S_k \le j-1\}$ is $(X_0, \dots, X_{j-1})$-meas.

(6) $\mathbb{P}[U < \infty] = 1$

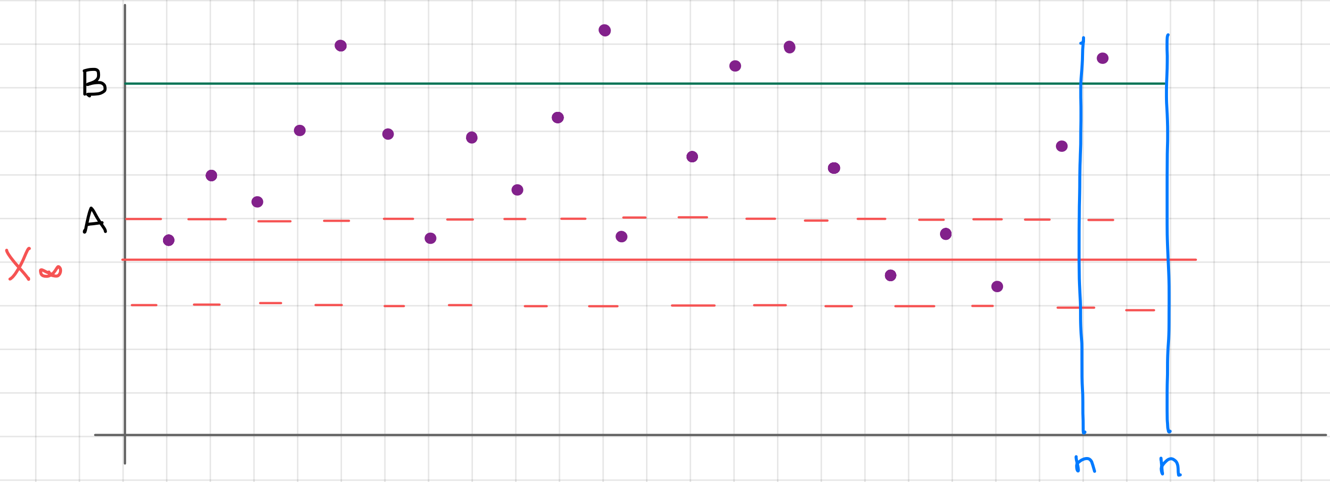
- Consider the following game:

bet $B_j = \begin{cases} 1, & S_k < j \le T_k \\ 0, & T_k < j \le S_{k+1} \end{cases}$, win/lose $B_j(X_j - X_{j-1})$

$(X_0, \dots, X_{j-1})$-measurable

Total winnings: $W_n = \sum_{j=1}^{n} B_j (X_j - X_{j-1})$

The martingale convergence theorem



- $(W_n)_{n\geq 1}$ is a martingale, therefore
  $\mathbb{E}[W_n] = \mathbb{E}[W_1] = 0$: $W_1 = \begin{cases} 0, & \text{if } X_0 > A \\ X_1 - X_0, & X_0 \leq A \end{cases}$,

- $$W_n = \sum_{k:T_k\leq n} \sum_{S_k<j\leq T_k} 1\cdot(X_j - X_{j-1}) = \sum_{k=1}^{U_n} \sum_{S_k<j<T_k} (X_j - X_{j-1}) + \sum_{j=S_{U_n+1}}^{n} (X_j - X_{j-1})$$
  $$\geq U_n\cdot(B-A) + X_n - X_{S_{U_n}} \geq U_n(B-A) - A$$

The martingale convergence theorem

- $\mathbb{E}[W_n] = 0 \geq (B-A)\,\mathbb{E}[U_n] - A \implies \mathbb{E}[U_n] \leq \frac{A}{B-A} < \infty$

- $\lim_{n\to\infty} \mathbb{E}[U_n] = \mathbb{E}[U] < \infty \implies \mathbb{P}[U < \infty] = 1$

(7) For any $A, B \in \mathbb{Q}$, $A, B \geq 0$, $A < B$

$$\mathbb{P}[\text{ infinitely many } (A,B)\text{-upcrossings}] = 0$$

(8) $\mathbb{P}[\exists\, A, B \in \mathbb{Q},\ A, B \geq 0,\ A < B$ s.t. the exists ∞-many (A,B)-upcrossings$] = 0$

$$\mathbb{P}[\lim_{n\to\infty} X_n = X_\infty] = 1$$

$\blacksquare$

Example

$(X_n)_{n\geq 0}$ SSRW on $\mathbb{Z}$, $X_0 = 1$. $T = \min\{n \geq 0 : X_n = 0\}$

Consider $M_n := X_{T\wedge n}$. M_n is a nonnegative martingale.

Therefore, by the Martingale convergence thm there exists r.v. M_∞ s.t. $\mathbb{P}\left[\lim_{n\to\infty} M_n = M_\infty\right] = 1$.

What is M_∞? $M_n(\omega)$ is eventually constant for any ω. Since $\{M_n(\omega) = k,\ M_{n+1}(\omega) = k\}$ is not possible for any $k \geq 1$, $M_\infty = 0$ with probability 1.

Remark $\mathbb{E}[M_n] = \mathbb{E}[M_0] = \mathbb{E}[X_0] = 1$, but $M_\infty = 0$.

In particular, $\lim_{n\to\infty} \mathbb{E}[M_n] \neq \mathbb{E}[\lim_{n\to\infty} M_n]$

Example. Polya Urns

An urn initially contains a red balls and b blue balls. At each step, draw a ball uniformly at random and return it with another ball of the same color. Denote by X_n the number of red balls in the urn after n turns. Then (X_n) is a Markov chain (time inhomogeneous)

$$\mathbb{P}[X_{n+1} = k+1 \mid X_n = k] = \frac{k}{n+a+b} \quad , \quad \mathbb{P}[X_{n+1} = k \mid X_n = k] = 1 - \frac{k}{n+a+b}$$

Long-run behavior of the process? Techniques developed for time-homogeneous MC cannot be applied.

Let $M_n := \dfrac{X_n}{n+a+b}$ be the fraction of red ball after n turns.

Then $0 \leq M_n \leq 1$, $\mathbb{E}[|M_n|] \leq 1$

Example. Polya Urns

Next,
$$\mathbb{E}[X_{n+1} | X_0, \dots, X_n] = \mathbb{E}[X_{n+1} | X_n] \quad (X_n \text{ is Markov})$$

and
$$\mathbb{E}[X_{n+1} | X_n] = (X_n+1) \cdot \frac{X_n}{n+a+b} + X_n \left(1 - \frac{X_n}{n+a+b}\right)$$
$$= \frac{X_n}{n+a+b} + X_n = X_n \frac{n+1+a+b}{n+a+b}$$

$$\mathbb{E}[M_{n+1} | M_0, \dots, M_n] = \mathbb{E}\left[\frac{X_{n+1}}{n+1+a+b} \Big| X_0, \dots, X_n\right] = \frac{X_n}{n+a+b} = M_n$$

(M_n) is a nonnegative martingale. Therefore, by the Martingale convergence theorem $M_n \to M_\infty$, $n \to \infty$ a.s.

One can show that M_∞ has beta distribution
$$f_{M_\infty}(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1$$