MATH 285: Stochastic Processes

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Today: Martingales.
Doob's maximal inequality

- Homework 6 is due on Friday, March 4, 11:59 PM

Martingales

Def 24.1 A discrete-time martingale is a stochastic process $(X_n)_{n\geq 0}$ which satisfies $\mathbb{E}[|X_n|]<\infty$ and

$$\mathbb{E}[X_{n+1} \mid X_{0,\dots,} X_n] = X_n \quad \text{for all } n\geq 0$$

Thm 24.8 (Optional sampling theorem)

Let $(X_n)_{n\geq 0}$ be a martingale, and let T be a finite stopping time. Suppose that either

(1) T is bounded: $\exists\, N<\infty$ s.t. $\mathbb{P}[T\leq N]=1$; or

(2) $(X_n)_{0\leq n\leq T}$ is bounded: $\exists\, B<\infty$ s.t. $\mathbb{P}[\,|X_n|\leq B \text{ for all } n\leq T]=1$

Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Example

Example 25.1 Let (X_n) be a SSRW on $\mathbb{Z}$ conditioned to start at $X_0 = j$ for some $j \in \{0, \dots, N\}$. (X_n) is a martingale.

Denote $\tau_k :=$ $T =$ (stopping times).

We computed using the first-step analysis.

Another approach: use the optional sampling theorem.

- (X_n) is a martingale
- $0 \le X_n \le N$ for all $0 \le n \le T$

By the Optional sampling theorem

X_T takes two values, so $\mathbb{E}[X_T] =$

so $\mathbb{P}[X_T = N] =$, $\mathbb{P}[X_T = 0) =$. Finally,

$\mathbb{P}[X_T = N] =$, $\mathbb{P}[X_T = 0] =$

Example

Let $X_1, \dots, X_n, \dots$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_n|] < \infty$, $\mathbb{E}[X_n] = \mu$ for all n, and denote $S_n := X_1 + \cdots + X_n$

and

Then $\mathbb{E}[|M_n|] \leq$

$\mathbb{E}[M_{n+1} | M_0, \dots, M_n] =$

$=$

$\mathbb{E}[M_1 | M_0] =$. (M_n) is a martingale.

Let T be a bounded stopping time for (X_n) (and for (M_n)).

Then by the Optional sampling theorem

Therefore,

Submartingales / supermartingales

A stochastic process (X_n) is called

a submartingale if $\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] \geq X_n$ for all n

a supermartingale if $\mathbb{E}[X_{n+1} \mid X_0, \dots, X_n] \leq X_n$ for all n

We use (sub) martingales to establish the maximal inequalities. Recall the Markov's inequality : $\forall a > 0$

In particular, if (X_n) is a submartingale and $X_n \geq 0$, then for any $i \leq n$ $\quad \mathbb{P}[X_i \geq a] \leq$

In fact a stronger statement holds.

Doob's maximal inequality

Thm 25.3 Let (X_n) be a non-negative submartingale.

Then for any $a>0$

Proof. Let $T :=$, a stopping time.

- $A_k := \{T = k\} =$
- Since $X_n \geq 0$, $\mathbb{E}[X_n] \geq$
- $\mathbb{E}[X_n \mathbb{1}_{A_k}] =$
- $\mathbb{E}[X_n] \geq$
- $\mathbb{P}[T \leq n] =$

Doob's maximal inequality

Lemma 25.4 Let (X_n) be a martingale, and let $f: \mathbb{R} \to \mathbb{R}$ be a such that $\mathbb{E}[|f(X_n)|] < \infty$ for all n.

Then

Proof Exercise.

Corollary 25.5 Let (X_n) be a martingale, let $r \geq 1$, $a, b \geq 0$.

Then

(i) $\mathbb{P}[\max\{X_0, \dots, X_n\} \geq a] \leq$

(ii) $\mathbb{P}[\max\{X_0, \dots, X_n\} \geq a] \leq$

Proof. If $r \geq 1$, then $f(x) = |x|^r$ is a convex function.

By Lemma 25.4 $(|X_n|^r)$ is a non-negative submartingale.

Doob's maximal inequality

Fix $a > 0$. If $X_k \geq a$, then Therefore,

$$\mathbb{P}[\max\{X_0, \dots, X_n\} \geq a] \leq$$

$$\leq$$

The second inequality is proven using a similar argument. ∎

Example 25.6 Let $X_1, X_2, \dots$ be i.i.d. symmetric Bernoulli random variables, . (S_n) is a martingale.

Take (ii) in Corollary 25.5 with $b =$ $a =$, so that

$$\mathbb{P}[\max\{S_0, \dots, S_n\} \geq \alpha\sqrt{n}] \leq$$

Now $\mathbb{E}[e^{S_n/\sqrt{n}}] =$

Therefore, for any n