MATH 285: Stochastic Processes

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Today: Martingales

- Homework 6 is due on Friday, March 4, 11:59 PM

Martingales

Def 24.1 A discrete-time martingale is a stochastic process $(X_n)_{n\geq 0}$ which satisfies and

Lemma 24.2 If $(X_n)_{n\geq 0}$ is a martingale, then

for all $m<n$.

Proof. Fix m. Induction. Holds for $n=m$, $n=m+1$.

Suppose for some $n>m$. Then

by the Tower property

$\mathbb{E}[X_{n+1} | X_0, \dots, X_m] =$

$=$

Martingales

Corollary 24.3 If $(X_n)_{n\geq 0}$ is a martingale, then it has constant expectation: for all n.

Proof. Use the double expectation property

Example (Betting on independent coin tosses)

Consider a game: bet dollar and toss a coin.

$$X_i = \begin{cases} 1, & \text{if you win the } i\text{-th toss} \\ -1, & \text{if you lose the } i\text{-th toss} \end{cases}, \quad X_1, X_2, \dots \text{ independent}$$

Denote by W_0 the initial fortune, independent of $X_1, X_2, \dots$

Let $W_n =$. We call $B_1, B_2, \dots$ the betting strategy.

Betting on independent coin tosses

Then

$$\mathbb{E}[\,|W_n|\,] \leq$$

and

$$\mathbb{E}[\,W_{n+1} \mid W_0, \ldots, W_n\,] =$$

$$=$$

$$=$$

Since $W_k = \sum_{i=1}^{k} X_i B_i$,

- W_0 is independent of X_{n+1}
- B_1 is W_0 measurable, $W_1 = W_0 + X_1 B_1$
- B_2 is (W_0, W_1) measurable, $W_2 = W_1 + X_2 B_2$ ...

Then $\mathbb{E}[\,X_{n+1} \mid W_0, \ldots, W_n\,] =$, and

i.e., W_n is a martingale

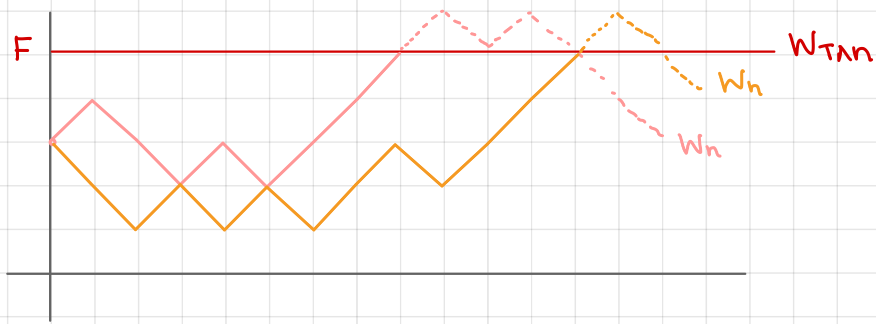
Stopping times

Let $(W_n)_{n\geq 0}$ be a stochastic process. Recall that random variable $T \in \{0,1,2,\dots\} \cup \{\infty\}$ is a stopping time if the fact that $\{T \leq n\}$ holds can be determined from $W_0, \dots, W_n$

Example of a stopping times: the first time the process hits some set/value, $T =$

Suppose you stop the game as soon as your fortune gets $\geq F$. Then the original process (W_n) is replaced by ,

where $T \wedge n =$



$W_{T\wedge n} = \Big\{$

Stopped martingale

Prop. 24.5 Let $(X_n)_{n\geq 0}$ be a martingale and let T be a stopping time for this martingale. Then

is a martingale.

Proof. Denote $Y_n :=$. Then $Y_n = X_{T\wedge n} \in \{X_0, X_1, \dots, X_n\}$ for each n, so $|Y_n| \leq$, and

$\mathbb{E}[|Y_n|] \leq$

Now we need to show that $\mathbb{E}[Y_{n+1} \mid Y_0, \dots, Y_n] = Y_n$.

(1) $\mathbb{E}[Y_{n+1} \mid X_0, \dots, X_n] = Y_n$

- $Y_{n+1} = X_{T\wedge(n+1)} =$
- $\{T \leq n\}$ only depends on $X_0, \dots, X_n$, so $\mathbb{1}_{\{T\leq n\}}$ is

  and is $(X_0, \dots, X_n)$ measurable

Stopped martingale

- $\mathbb{1}_{\{T>n\}} =$
- Using the properties of the conditional expectation

$$\mathbb{E}[Y_{n+1} \mid X_0, \dots, X_n] = \mathbb{E}[X_T \mathbb{1}_{\{T\leq n\}} \mid X_0, \dots, X_n] + \mathbb{E}[X_{n+1} \mathbb{1}_{\{T>n\}} \mid X_0, \dots, X_n]$$

$$=$$

$$=$$

- $X_T \mathbb{1}_{\{T\leq n\}} + X_n \mathbb{1}_{\{T>n\}} =$

(2) $\mathbb{E}[Y_{n+1} \mid Y_0, \dots, Y_n] = Y_n$

- $\bar{Y}_n := (Y_0, \dots, Y_n)$ is $\bar{X}_n := (X_0, \dots, X_n)$ measurable
- Using the Tower property $\mathbb{E}[Y_{n+1} \mid \bar{Y}_n] =$
- By (1) $\mathbb{E}[\mathbb{E}[Y_{n+1} \mid \bar{X}_n] \mid \bar{Y}_n] =$

Martingale betting strategy

Corollary For any $n \in \mathbb{N}$ $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_{T\wedge 0}] = \mathbb{E}[X_0]$

Example (Martingale betting strategy)

Consider the betting strategy $B_n = 2^{n-1}$ (double each round).

Let $T = \min\{n : X_n = 1\}$, the time of the first win. If $W_0 = C$,

then

$$\mathbb{E}[W_T] =$$

The event $T = n$ corresponds to a specific trajectory

$X_1 = -1, X_2 = -1, \dots, X_{n-1} = -1, X_n = 1$, so $\mathbb{P}[T=n] =$

So $\mathbb{E}[W_T] = \sum_{n=1}^{\infty} \mathbb{E}[W_n \mid X_1 = -1, \dots, X_{n-1} = -1, X_n = 1] \frac{1}{2^n}$

$=$

Problem T can be arbitrarily large.

Optional Sampling Theorem

Thm 24.8 Let $(X_n)_{n\geq 0}$ be a martingale, and let T be a finite stopping time. Suppose that either

(1) T is bounded : $\exists\, N < \infty$ s.t. ; or

(2) $(X_n)_{0\leq n\leq T}$ is bounded : $\exists\, B < \infty$ s.t.

Then $\mathbb{E}[X_T] =$

Proof.

- Suppose (1) holds. By Prop. 24.5 $X_{T\wedge n}$ is a martingale, $\mathbb{E}[X_{T\wedge n}] =$ for all n.

  Then $\mathbb{E}[X_0] =$.

- Suppose that (2) holds (T is not necessarily bounded)

  Then $X_T =$

  First term: $\mathbb{E}[X_{T\wedge n}] =$

Optional Sampling Theorem

Second term:

$$\left|\mathbb{E}\left[(X_T - X_{T\wedge n})\mathbb{1}_{\{T>n\}}\right]\right| \leq$$

$$\leq$$

Since $\mathbb{P}[T<\infty]=1$,

Therefore,

$$\left|\mathbb{E}[X_T]-\mathbb{E}[X_0]\right| =$$