MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

Today: Long-run behaviour of continuous time MC

Martingales. Conditional expectation

- Homework 6 is due on Friday, March 4, 11:59 PM

Convergence to the stationary distribution

The exact analog of the convergence theorems for discrete time MC (Cor. 11.1, Thm 11.3, Thm 12.1)

<u>Thm 22.8</u> Let (X_t) be an irreducible, continuous time MC with transition rates $q(i,j)$. Then TFAE:

(1) All states are positive recurrent

(2) Some state is positive recurrent

(3) The chain is non-explosive and there exists a stationary distribution π.

Moreover, when these conditions hold, the stationary distribution is given by $\pi(j) = \dfrac{1}{\mathbb{E}_j[T_j]}$, where T_j is the return time to j; and $\lim_{t\to\infty} P_t(i,j) = \pi(j)$ for any states i,j.

Convergence to the stationary distribution

<u>Remark</u> There is no issue with periodicity: if $p_t(i,j)>0$ for some $t>0$, then $p_t(i,j)>0$ for all $t>0$

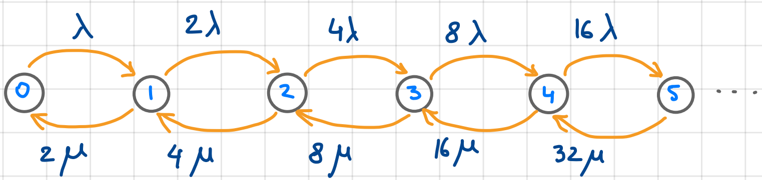
Example: M/M/1 queue is positive recurrent if $\lambda<\mu$

null recurrent if $\lambda=\mu$

transient if $\lambda>\mu$

M/M/∞ queue is always positive recurrent

Example:



$\lambda_j = \lambda 2^j$

$\mu_j = \mu 2^j$

$$\theta_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} = \frac{\lambda \cdot 2\lambda \cdot \dots \cdot 2^{j-1}\lambda}{2\mu \cdot 4\mu \cdot \dots \cdot 2^j \mu} = \left(\frac{\lambda}{\mu}\right)^j \frac{1}{2^j}.$$

If $\frac{\lambda}{\mu} \in (1,2)$, then $\sum_{j\geq 0}^{\infty} \theta_j < \infty$

but the explosion occurs.

Martingales

Motivating example

Consider a game: bet 1 dollar and toss a coin.

$$B_i = \begin{cases} 1, & \text{if you win the } i\text{-th toss} \\ -1, & \text{if you lose the } i\text{-th toss} \end{cases}$$

Let X_n be your total winning after n tosses

$$X_n = B_1 + B_2 + \cdots + B_n \qquad (\text{SSRW on } \mathbb{Z},\ X_0 = 0)$$

Then for any $n \in \mathbb{N}$ $\quad \mathbb{E}[X_n] = \sum_{i=1}^{n} \mathbb{E}[B_i] = 0 \quad$ (fair game)

Suppose that you observed n tosses. What can you say about the expected winnings at time $n+1$ given that you know the trajectory of X up to time n?

Motivating example

For a SSRW on $\mathbb{Z}$ the answer is trivial:

$$\mathbb{E}[X_{n+1} \mid X_0 = i_0, X_1 = i_1, \dots, X_n = i_n] \overset{MP}{=} \mathbb{E}[X_{n+1} \mid X_n = i_n]$$

$$= \mathbb{E}[X_n + B_{n+1} \mid X_n = i_n] = i_n + \mathbb{E}[B_{n+1}] = i_n$$

Similarly, for any $m \in \mathbb{N}$

$$\mathbb{E}[X_{n+m} \mid X_0 = i_0, \dots, X_n = i_n] = i_n + \mathbb{E}[B_{n+1}] + \dots + \mathbb{E}[B_{n+m}] = i_n$$

or written in a different form

$$\mathbb{E}[X_{n+m} - X_n \mid X_0 = i_0, \dots, X_n = i_n] = 0$$

No matter what has happened to the player's fortute so far, the expected net win or loss for any future time is always zero. We call such processes martingales.

Conditional expectation

Let X be a (discrete) random variable, $X \in S \subset \mathbb{R}$, and let B be an event. Then the conditional expectation is given by $\mathbb{E}[X \mid B] = \sum_{x \in S} s \cdot \mathbb{P}[X = x \mid B]$

Often B has the form $B = \{Y_1 = i_1, Y_2 = i_2, \dots, Y_n = i_n\}$

We can group all these events into a new random variable

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] := \sum_{i_1, \dots, i_n} \mathbb{E}[X \mid Y_1 = i_1, \dots, Y_n = i_n] \cdot \mathbb{1}_{\{Y_1 = i_1, \dots, Y_n = i_n\}}$$

Think in the following way: Start with random variable X; then we are given some information in the form of random variables $Y_1, \dots, Y_n$ that we may observe. Then $\mathbb{E}[X \mid Y_1, \dots, Y_n]$ is our best guess about the value of X given $Y_1, \dots, Y_n$ (as a function of $Y_1, \dots, Y_n$)

Examples

Suppose that $X = F(Y_1, \dots, Y_n)$. X is completely determined by $Y_1, \dots, Y_n$. What is the best guess for the value of X given $Y_1, \dots, Y_n$? X itself.

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] = \mathbb{E}[F(Y_1, \dots, Y_n) \mid Y_1, \dots, Y_n]$$

$$= \sum_{i_1, \dots, i_n} \mathbb{E}[F(Y_1, \dots, Y_n) \mid Y_1 = i_1, \dots, Y_n = i_n] \mathbb{1}_{\{Y_1 = i_1, \dots, Y_n = i_n\}}$$

$$= \sum_{i_1, \dots, i_n} F(i_1, \dots, i_n) \mathbb{1}_{\{Y_1 = i_1, \dots, Y_n = i_n\}} = F(Y_1, \dots, Y_n) = X$$

When X is a function of $Y_1, \dots, Y_n$, we say that

X is measurable with respect to $Y_1, \dots, Y_n$

Conclusion: If X is measurable with respect to $Y_1, \dots, Y_n$, then

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] = X$$

Examples

Another extreme situation. Suppose that X and $Y_1, \dots, Y_n$ are independent. This means that any information about $Y_1, \dots, Y_n$ should be essentially useless in determining the value of X, the best guess is simply $\mathbb{E}[X]$. Indeed for any $i_1, \dots, i_n$

$$\mathbb{E}[X \mid Y_1 = i_1, \dots, Y_n = i_n] = \sum_x x \, \mathbb{P}[X = x \mid Y_1 = i_1, \dots, Y_n = i_n] = \sum_x x \, \mathbb{P}[X = x] = \mathbb{E}[X]$$

Thus

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] = \sum_{i_1, \dots, i_n} \mathbb{E}[X \mid Y_1 = i_1, \dots, Y_n = i_n] \, \mathbb{1}_{\{Y_1 = i_1, \dots, Y_n = i_n\}} = \sum_{i_1, \dots, i_n} \mathbb{E}[X] \, \mathbb{1}_{\{Y_1 = i_1, \dots, Y_n = i_n\}}$$

$$= \mathbb{E}[X]$$

Conclusion: If X and $Y_1, \dots, Y_n$ are independent, then

$$\mathbb{E}[X \mid Y_1, \dots, Y_n] = \mathbb{E}[X]$$

Examples

Let X_n be a SSRW on $\mathbb{Z}$. Then

$$\mathbb{E}[X_{n+m} - X_n \mid X_0, \dots, X_n] = \sum_{i_0, \dots, i_n} \mathbb{E}[X_{n+m} - X_n \mid X_0 = i_0, \dots, X_n = i_n] \mathbb{1}_{\{X_0 = i_0, \dots, X_n = i_n\}}$$

$$= 0$$

Also, $\mathbb{E}[X_n \mid X_0, \dots, X_n] = X_n$. Therefore,

$$\mathbb{E}[X_{n+m} - X_n \mid X_0, \dots, X_n] = \mathbb{E}[X_{n+m} \mid X_0, \dots, X_n] - \mathbb{E}[X_n \mid X_0, \dots, X_n]$$

$$= \mathbb{E}[X_{n+m} \mid X_0, \dots, X_n] - X_n = 0$$

and $\mathbb{E}[X_{n+m} \mid X_0, \dots, X_n] = X_n$

The best guess about our future fortune is our present fortune, the "average fairness" that difines martingales.

Properties of conditional expectation

Prop 23.5

Let X, X' be random variables, and $\bar{Y} = \{Y_1, \dots, Y_n\}$ a collection of random variables. Then the following holds:

(1) For $a, b \in \mathbb{R}$, $\mathbb{E}[aX + bX' \mid \bar{Y}] = a\,\mathbb{E}[X \mid \bar{Y}] + b\,\mathbb{E}[X' \mid \bar{Y}]$

(2) If X is $\bar{Y}$- measurable, then $\mathbb{E}[X \mid \bar{Y}] = X$

(3) If X is independent of $\bar{Y}$, then $\mathbb{E}[X \mid \bar{Y}] = \mathbb{E}[X]$

(4) (Tower property) Let $\bar{Z} = \{Z_1, \dots, Z_m\}$ be another collection of random variables, and suppose that $\bar{Y}$ is $\bar{Z}$ measurable, $\bar{Y} = F(\bar{Z})$ (typical situation $\underset{\{Y_1, \dots, Y_n, Y_{n+1}\}}{\underset{\shortparallel}{\bar{Z}}} \supseteq \bar{Y}$). Then

$$\mathbb{E}\big[\mathbb{E}[X \mid \bar{Z}] \mid \bar{Y}\big] = \mathbb{E}[X \mid \bar{Y}]$$

(5) (Factoring) If Y is $\bar{Y}$-measurable, then $\mathbb{E}[XY \mid \bar{Y}] = Y\,\mathbb{E}[X \mid \bar{Y}]$

Properties of conditional expectation

<u>Cor 23.6</u> Particular case of the Tower property

$$\mathbb{E}[\,\mathbb{E}[X \mid \bar{Y}]\,] = \mathbb{E}[X]$$

<u>Proof</u>. Take $\bar{Z} = \emptyset$. Then $\bar{Z}$ is independent of any collection of random variables, and $\bar{Y} \supset \emptyset$. Thus by the tower property

$$\mathbb{E}[\,\mathbb{E}[X \mid \bar{Y}] \mid \emptyset] \overset{(4)}{=} \mathbb{E}[X \mid \emptyset] \overset{(3)}{=} \mathbb{E}[X]$$

and

$$\mathbb{E}[\,\mathbb{E}[X \mid \bar{Y}] \mid \emptyset] \overset{(3)}{=} \mathbb{E}[\,\mathbb{E}[X \mid \bar{Y}]\,] \qquad \blacksquare$$