MATH 285: Stochastic Processes

math-old.ucsd.edu/~ynemish/teaching/285

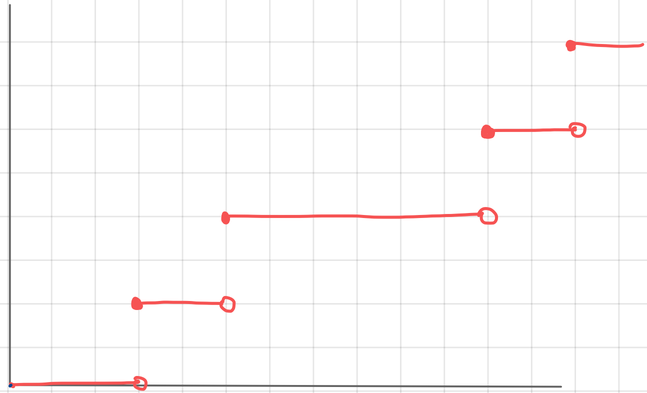
Today: Poisson processes
Birth and death chains
Recurrence and transience

- Homework 6 is due on Friday, March 4, 11:59 PM

Poisson processes

! The jump chain of a Poisson process has a deterministic trajectory

$$Y_n = Y_0 + n$$



By Prop. 19.2, given the trajectory the sojourn time are independent exponential r.v. with $S_k \sim Exp(q(Y_{k-1}))$

$$\mathbb{P}[S_1>s_1,\dots,S_n>s_n] = \sum_{i_0,\dots,i_n} \mathbb{P}[S_1>s_1,\dots,S_n>s_n \mid Y_0=i_0,\dots,Y_n=i_n]\,\mathbb{P}[Y_0=i_0,\dots,Y_n=i_n]$$

$$= \mathbb{P}[S_1>s_1,\dots,S_n>s_n \mid Y_0=i_0,\ Y_1=i_0+1,\dots,Y_n=i_0+n]\,\mathbb{P}[Y_0=i_0,\ Y_1=i_0+1,\dots]$$

$$= e^{-q(i_0)s_1}\, e^{-q(i_0+1)s_2}\cdots e^{-q(i_0+n-1)s_n}\cdot \mathbb{P}[Y_0=i_0] = e^{-\lambda s_1}\cdots e^{-\lambda s_n}\,\mathbb{P}[Y_0=i_0]$$

Prop 20.6 If (X_t) is a Poisson process, then $S_1, S_2, \dots$ are i.i.d with $S_1 \sim Exp(\lambda)$

Poisson processes

Alternative construction of a Poisson process (with $X_0 = 0$):

- take a collection of i.i.d. random variables S_k, $S_k \sim \mathrm{Exp}(\lambda)$
- define the jump times $J_n = S_1 + \dots + S_n$, $J_0 = 0$
- set $X_t = n$ for $J_n \leq t < J_{n+1}$

Then X_t is a Poisson process with rate λ.

You can think about J_n as the times of some events, and X_t as the number of events that happend up to time t.

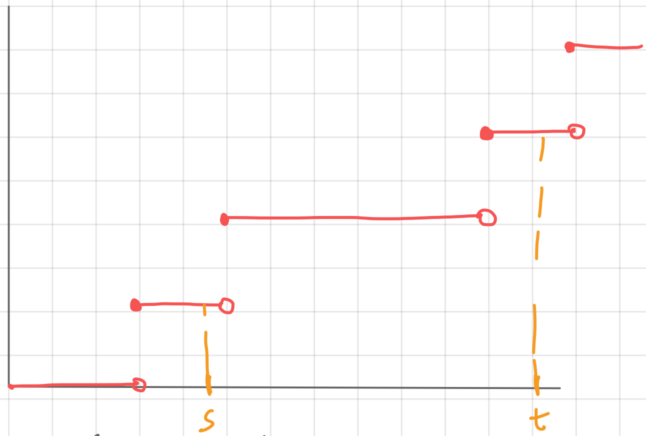
<u>Theorem 20.7</u> Let $(X_t)_{t \geq 0}$ be a Poisson process of rate λ, $X_0 = 0$. Then for any $s \geq 0$ the process $\tilde{X}_t = X_{t+s} - X_s$ is a Poisson process of rate λ, independent of $\{X_u : 0 \leq u \leq s\}$

No proof.

Independent increments

Given a stochastic process $(X_t)_{t\geq 0}$ its increments are random variables

$$X_t - X_s, \quad 0 \leq s < t < \infty$$



Suppose that (X_t) is a counting process, i.e., $\mathbb{P}[X_{J_{n+1}} = i+1 \mid X_{J_n} = i]$ (jump times = event times, X_t = # of events that occurred up to time t). Then for $s<t$ $X_t - X_s$ = # of events that occurred on $(s, t]$.

Cor. 20.8 If (X_t) is a Poisson process with rate λ, then for any $0 \leq t_0 < t_1 < \cdots < t_n$ the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent, and each increment $X_t - X_s$ is a Poisson random variable with rate $\lambda(t-s)$. These properties uniquely characterize the Poisson process.

Independent increments

Proof. • $X_t - X_s = X_{s+(t-s)} - X_s \sim \text{Pois}(\lambda(t-s))$ [by Thm 20.7]

• $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

Induction: Suppose $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent

By Thm 20.7, for any $t \geq 0$ the process

$\tilde{X}_t := X_{t_n+t} - X_{t_n}$ is independent of X_s for $s \leq t_n$

Therefore, $\tilde{X}_t$ is independent of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$,

and for any $t_{n+1} > t_n$ $\tilde{X}_{t_{n+1}-t_n} = X_{t_{n+1}} - X_{t_n}$ is independent

of $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$

• Independent increments uniquely determine the joint distribution of $(X_{t_0}, \dots, X_{t_n})$ for any $0 \leq t_0 < \dots < t_n < \infty$

$$\mathbb{P}[X_{t_0} = i_0, \overset{X_{t_1}=i_1}{\dots}, X_{t_n} = i_n] = \mathbb{P}[X_{t_0} - X_0 = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \dots, X_{t_n} - X_{t_{n-1}} = i_n - i_{n-1}]$$

$$= \mathbb{P}[X_{t_0} - X_0 = i_0] \cdots \mathbb{P}[X_{t_n} - X_{t_{n-1}} = i_n - i_{n-1}] \quad \blacksquare$$

Birth and death chains

Consider a continuous-time MC with state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(i, i+1) = \lambda_i \geq 0, \quad q(i, i-1) = \mu_i \geq 0, \quad q(i,j) = 0 \text{ if } j \notin \{i \pm 1\}$$

We call this process the birth and death chain.

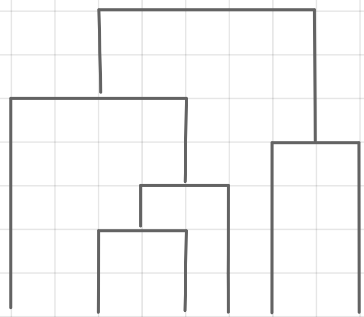
- all $\mu_i = 0$ pure birth process
- all $\lambda_i = 0$ pure death process
- Poisson process is a pure birth process with $\lambda_i = \lambda$

Example : Kingman's coalescent

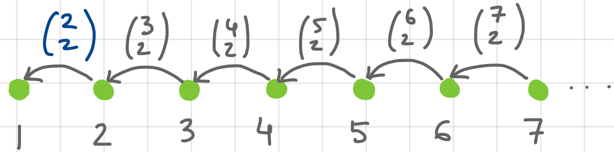
Pure death process with $\mu_1 = 0$, $\mu_k = \binom{k}{2}$

Tracking ancestor lines back in time

$$\mathbb{E}[\min\{t \geq 0 : X_t = 1\}]$$



Kingman's coalescent



Denote $T: \min\{t \geq 0 : X_t = 1\}$

the time to most recent common ancestor.

Conditioned on $X_0 = N$, $T = S_1 + S_2 + \cdots + S_{N-1}$, where

S_1 = time spent at state N, S_2 = time spent at $N-1$, ...

$S_1 \sim \mathrm{Exp}\left(\binom{N}{2}\right)$, $S_2 \sim \mathrm{Exp}\left(\binom{N-1}{2}\right)$ . . .

$$\mathbb{E}[T] = \mathbb{E}[S_1 + S_2 + \cdots + S_{N-1}] = \frac{1}{\binom{N}{2}} + \frac{1}{\binom{N-1}{2}} + \cdots + \frac{1}{\binom{2}{2}}$$

$$= \frac{2}{N(N-1)} + \frac{2}{(N-1)(N-2)} + \cdots + \frac{2}{2} = 2\left[\frac{1}{N-1} - \frac{1}{N} + \frac{1}{N-2} - \frac{1}{N-1} + \cdots + \frac{1}{1} - \frac{1}{2}\right] = 2\left[1 - \frac{1}{N}\right]$$

Denote L = sum of the branch lengths. Compute

Conditioned on $X_0 = N$, $L = N \cdot S_1 + (N-1) \cdot S_2 + \cdots + 2 \cdot S_{N-1}$

$$\mathbb{E}[L] = N \frac{2}{N(N-1)} + (N-1) \cdot \frac{2}{(N-1)(N-2)} + \cdots + 2 \cdot \frac{2}{2} = 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{N-1}\right) \approx 2 \log N$$

Explosion

Let (X_t) be a pure birth process with $\lambda_i = i^2$.

Condition on $X_0 = 1$. Denote by T_N the time to reach N.

Then $T_N = S_1 + S_2 + \cdots + S_{N-1}$ and

$$E[T_N] = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(N-1)^2}$$

Denote $T := \sum_{i=1}^{\infty} S_i$ the time to reach infinity. Then

$$E[T] = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} < \infty,$$ and thus $P[T < \infty] = 1$

We call T the explosion time.

What happens after T?

We can set $X_t = \infty$ for $t \geq T$ (minimal)

or we can restart from another state