MATH 285: Stochastic Processes

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Today: Strong Markov property
Embedded jump chain
Infinitesimal description

- Homework 5 is due on Sunday, February 20, 11:59 PM

Exponential distribution

We write $T \sim \text{Exp}(q)$. Here are some properties of exponential distribution

Prop. 18.3 Let $T_1, T_2, \dots, T_n$ be independent with $T_j \sim \text{Exp}(q_j)$

(a) Density $f_{T_j}(t) = q_j e^{-q_j t}$, $\mathbb{E}[T_j] = \frac{1}{q_j}$, $\text{Var}[T_j] = \frac{1}{q_j^2}$

(b) $\mathbb{P}[T_j > s+t \mid T_j > s] = \mathbb{P}[T_j > t]$

(c) $T = \min_j T_j$ is exponential with $T \sim \text{Exp}(q_1 + \cdots + q_n)$. Moreover

$$\mathbb{P}[T = T_j] = \frac{q_j}{q_1 + \cdots + q_n}$$

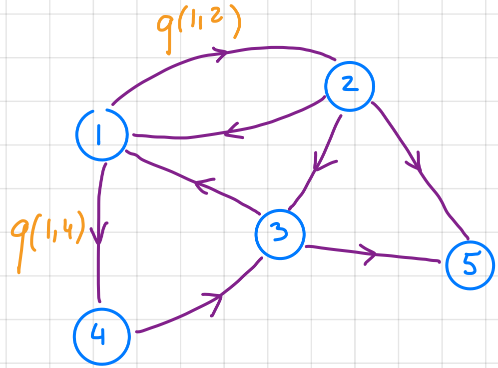
Proof. (a), (b) are trivial.

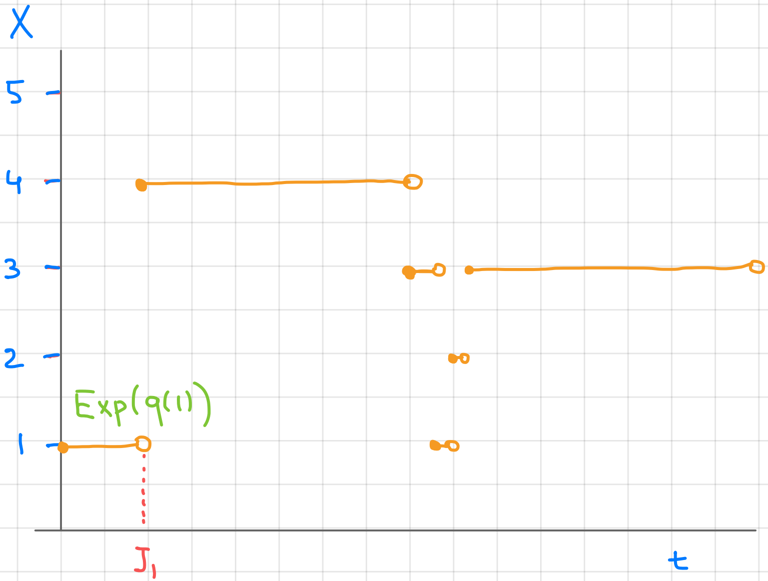
(c) $\mathbb{P}[T > t] = \mathbb{P}[T_1 > t, \dots, T_n > t] = \mathbb{P}[T_1 > t] \cdots \mathbb{P}[T_n > t] = e^{-q_1 t} \cdot \dots \cdot e^{-q_n t} = e^{-(q_1 + \cdots + q_n)t}$

$$\mathbb{P}[T = T_1] = \mathbb{P}[T_2 > T_1, \dots, T_n > T_1] = \int_0^\infty q_1 e^{-q_1 t} \, \mathbb{P}[T_2 > t, \dots, T_n > t] \, dt$$

$$= \int_0^\infty q_1 e^{-q_1 t} e^{-(q_2 + q_3 + \cdots + q_n)t} \, dt = \frac{q_1}{q_1 + q_2 + \cdots + q_n} \quad \blacksquare$$

Transition rates





- Conditioned on $X_0 = i$,

  $$J_1 \sim \mathrm{Exp}(q(i))$$

- Denote $p(i,j) = \mathbb{P}[X_{J_1} = j \mid X_0 = i]$

  $$p(i,i) = 0$$

- Define transition rates

  $$q(i,j) = q(i)\, p(i,j)$$

  $$q(i,j) \geq 0, \quad q(i,i) = 0$$

- $$\sum_j q(i,j) = q(i)$$

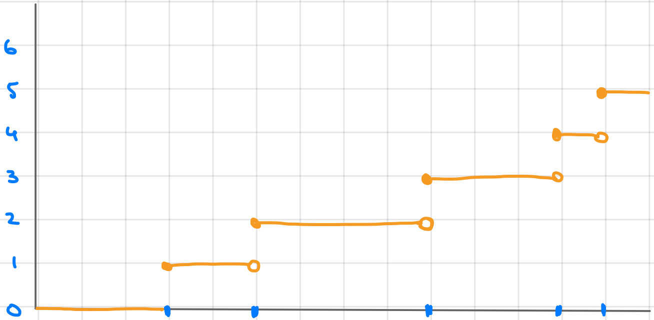
- $$p(i,j) = \frac{q(i,j)}{q(i)} = \frac{q(i,j)}{\sum_k q(i,k)}$$

Poisson process

Consider a continuous-time MC on the state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(i, i+1) = \lambda, \qquad q(i,j) = 0 \text{ for } j \neq i+1$$

We call this process the Poisson process with rate $\lambda > 0$.



Start a clock $\mathrm{Exp}(\lambda)$.

When it rings, move up.

Repeat ...

Proposition 18.5 Let $(X_t)_{t\geq 0}$ be a Poisson process with rate λ. The for any $t>0$, conditioned on $X_0 = 0$, $X_t \sim \mathrm{Pois}(\lambda t)$

$$\mathbb{P}[X_t = k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \in \mathbb{Z}_+$$

Strong Markov property

Given a MC $(X_t)_{t\geq 0}$, a stopping time T is a random variable taking values in $[0, +\infty]$ with property that

for $t \geq 0$ the event $\{T \leq t\}$ depends only on $\{X_s : s \leq t\}$

Thm 19.1 (Strong Markov property) Let $(X_t)_{t\geq 0}$ be a continuous-time MC with state space S and transition rates $q(i,j), i,j \in S$. Let T be a stopping time. For some $i>0$, suppose that $\mathbb{P}[X_T = i] > 0$. Then, conditioned on $X_T = i$, $(X_{T+t})_{t\geq 0}$ is a MC with the same transition rates $q(i,j), i,j \in S$, independent from $(X_t)_{0\leq t\leq T}$

No proof. Strong Markov property can be used to develop the first step analysis.

First step analysis

For any set $A \subset S$ denote the hitting time

$$\tau_A = \min\{t \geq 0 : X_t \in A\}$$

- For $A, B \subset S$, $A \cap B = \emptyset$, what is the probability of reaching A before B? $\mathbb{P}[\tau_A < \tau_B] = ?$

Set $h(i) = \mathbb{P}_i[\tau_A < \tau_B]$. Then

$$(*) \quad \begin{cases} h(i) = 1 & \text{if } i \in A \\ h(i) = 0 & \text{if } i \in B \\ h(i) = \sum_{j \in S} \frac{q(i,j)}{q(i)} h(j), & i \notin A \cup B \end{cases}$$

$$h(i) = \sum_{j \in S} \mathbb{P}_i[X_{J_1} = j]\, \mathbb{P}_i[\tau_A < \tau_B \mid X_{J_1} = j] = \sum_{j \in S} p(i,j)\, h(j)$$

$$\mathbb{P}_i[\tau_A < \tau_B \mid X_{J_1} = j] \overset{SMP}{=} \mathbb{P}_j[\tau_A - J_1 < \tau_B - J_1] = \mathbb{P}_j[\tau_A < \tau_B]$$

First step analysis

- Expected hitting time: $\mathbb{E}_i[\tau_A]$

Denote $g(i) := \mathbb{E}_i[\tau_A]$. Then

$$(**)\quad \begin{cases} g(i) = 0, & i \in A \\ q(i)g(i) = 1 + \sum_{j \in S} q(i,j) g(j), & i \notin A \end{cases}$$

$$g(i) = \sum_{j \in S} \mathbb{P}_i[X_{J_1} = j]\, \mathbb{E}_i[\tau_A \mid X_{J_1} = j] = \sum_{j \in S} \frac{q(i,j)}{q(i)} \mathbb{E}_i[\tau_A \mid X_{J_1} = j]$$

Define $Y_t = X_{J_1 + t}$. Then

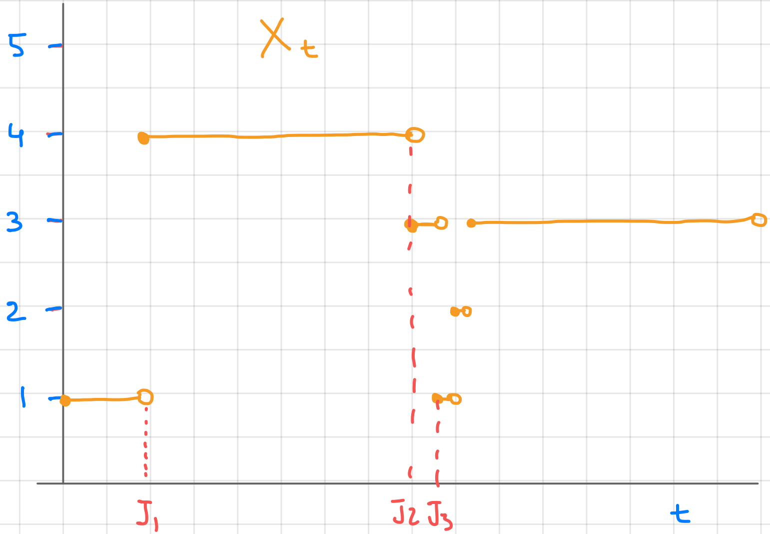
$$\tau_A = \min\{t \geq 0 : X_t \in A\} = \min\{t + J_1 : X_{J_1+t} \in A\}$$

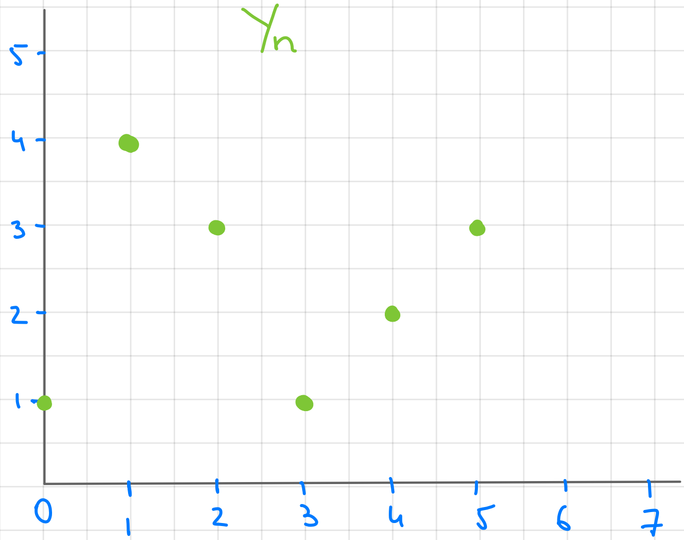
$$= \min\{t + J_1 : Y_t \in A\} = J_1 + \underbrace{\min\{t : Y_t \in A\}}_{=: \tilde{\tau}_A}$$

$$\mathbb{E}_i[\tau_A \mid X_{J_1} = j] = \mathbb{E}_i[J_1 + \tilde{\tau}_A \mid X_{J_1} = j] \overset{SMP}{=} \mathbb{E}_i[J_1] + \mathbb{E}_j[\tau_A] = \frac{1}{q(i)} + g(j)$$

$$\Rightarrow g(i) = \sum_{j \in S} \frac{q(i,j)}{q(i)} \left[\frac{1}{q(i)} + g(j)\right] \Rightarrow q(i)g(i) = 1 + \sum_{j \in S} q(i,j) g(j)$$

Embedded jump chain





Denote $J_0 = 0$, $J_{n+1} = \min\{t \geq J_n : X_t \neq X_{J_n}\}$

By the strong Markov property, for any $i_0, \dots, i_n \in S$

$$\mathbb{P}[X_{J_n}=i_n,\dots,X_{J_1}=i_1, X_0=i_0] = \mathbb{P}[X_{J_n}=i_n \mid X_{J_{n-1}}=i_{n-1},\dots,X_0=i_0]\,\mathbb{P}[X_{J_{n-1}}=i_{n-1},\dots,X_0=i_0]$$

$$= \mathbb{P}[X_{J_n}=i_n \mid X_{J_{n-1}}=i_{n-1}]\,\mathbb{P}[X_{J_{n-1}}=i_{n-1},\dots,X_0=i_0] = \cdots = p(i_{n-1},i_n)\cdots p(i_0,i_n)\,\mathbb{P}[X_0=i_0]$$

Denote $Y_n := X_{J_n}$, the embedded jump chain of $(X_t)_{t\geq 0}$.

Embedded jump chain

The embedded jump chain $(Y_n)_{n\geq 0}$ is a discrete-time MC with state space S and transition probabilities

$$\mathbb{P}[Y_1 = j \mid Y_0 = i] = \mathbb{P}[X_{J_1} = j \mid X_0 = i] = p(i,j) = \frac{q(i,j)}{q(i)}$$

What is the distribution of the time between two consecutive jumps? Denote by $S_k := J_k - J_{k-1}$ the sojourn times.

We know that $S_1 = J_1 \sim Exp(q(i_0))$. Denote $\tilde{X}_t := X_{J_{k-1}+t}$. Given $Y_{k-1} = i_{k-1}$ (and $J_{k-1} < \infty$) by the SMP for (X_t) and J_{k-1}, the first jump time of $\tilde{X}_t$ has exponential distribution $\tilde{J}_1 = J_k - J_{k-1} = S_k \sim Exp(q(i_{k-1}))$

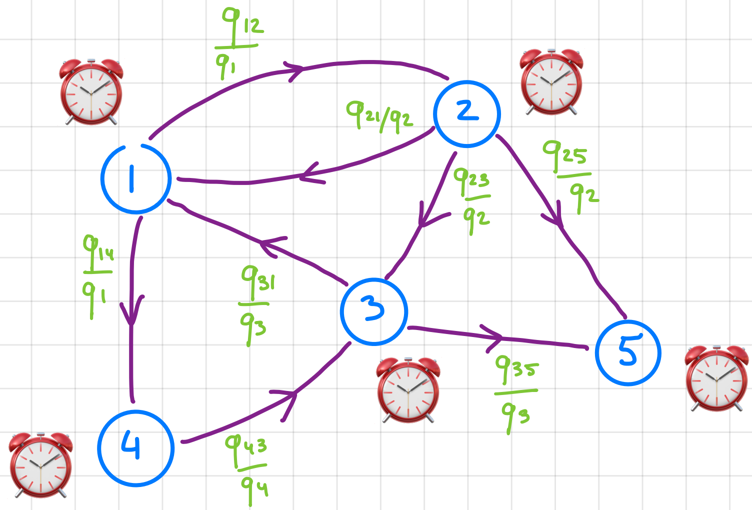
$\mathbb{P}[\tilde{X}_{\tilde{J}_1} = i_k] = \mathbb{P}[Y_k = i_k] = p(i_{k-1}, i_k)$, S_k, Y_k are indep. and indep. of $S_1, \dots, S_{k-1}$
$Y_0, \dots, Y_{k-1}$

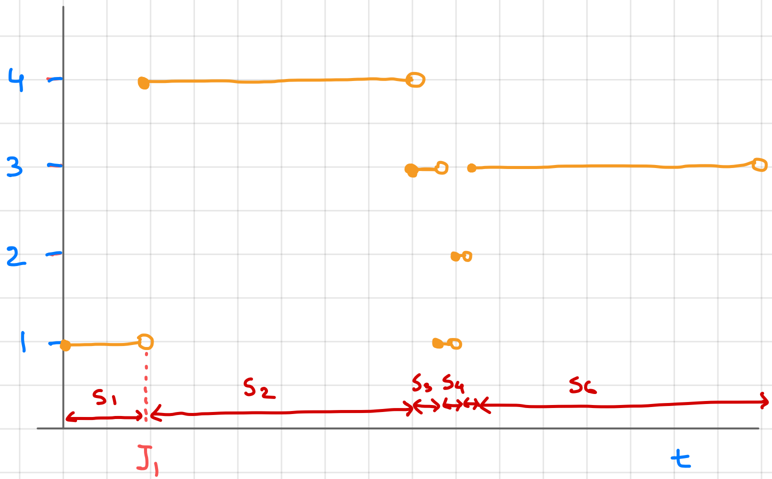
<u>Prop. 19.2</u> Conditioned on $Y_0, \dots, Y_{n-1}$, the sojourn times $S_1, \dots, S_n$ are independent exponential random variables with $S_k \sim Exp(q(Y_{k-1}))$

Embedded jump chain

Jump and hold construction

- embedded jump chain (Y_n) with $\mathbb{P}[\, Y_{n+1} = j \mid Y_n = i \,] = \dfrac{q_{ij}}{q_i}$
- exponential sojourn times S_n with $S_n \sim \mathrm{Exp}(q(Y_{n-1}))$





- start from $X_0 = Y_0 = i_0$
- wait at i_0 $\quad S_1 \sim \mathrm{Exp}(q(i_0))$
- $J_1 = S_1, \quad X_{J_1} = Y_1 = i_1$
- wait at i_1 $\quad S_2 \sim \mathrm{Exp}(q(i_1))$
- $J_2 = S_1 + S_2, \quad X_{J_2} = Y_2 = i_2$

$\vdots$

Infinitesimal description

Transition rates completely determine the Markov chain.

Q: What is the distribution of X_t? $\mathbb{P}_i[X_t = j] = p_t(i,j) = ?$

Thm 19.3 Let $(X_t)_{t\geq 0}$ be a MC with state space S and transition rates $q(i,j)$. Then the transition probabilities satisfy

$$p_t(i,i) = 1 - q(i)t + o(t) \quad \text{as } t \to 0 \quad \text{for } i \in S$$

$$p_t(i,j) = q(i,j)t + o(t) \quad \text{as } t \to 0 \quad \text{for } i \neq j$$

Proof.

(1) $p_t(i,i) = \mathbb{P}_i[X_t = i]$

(2) $p_t(i,j)$

$p_t(i,j) = \mathbb{P}_i[X_t = j] \geq$

$=$

$=$

Infinitesimal description

(3) We can write (1) and (2) as

$$p_t(i,i) \geq 1 - q(i)t + \xi_{ii}(t), \quad \xi_{ii}(t) = o(t)$$

$$p_t(i,j) \geq q(i,j)t + \xi_{ij}(t), \quad \xi_{ij}(t) = o(t)$$

Then

$$p_t(i,i) = 1 - q(i)t + \xi_{ii}(t)$$

$$p_t(i,j) = q(i,j)t + \xi_{ij}(t)$$

Take the sum

$$p_t(i,i) + \sum_{j \neq i} p_t(i,j) =$$

$\Rightarrow$ $\Rightarrow$ $\Rightarrow$

<u>Remark</u> In order to identify a Markov chain it is enough to compute $p_t(i,j)$ to first order in t as $t \to 0$.