MATH 285: Stochastic Processes

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Today: Branching processes

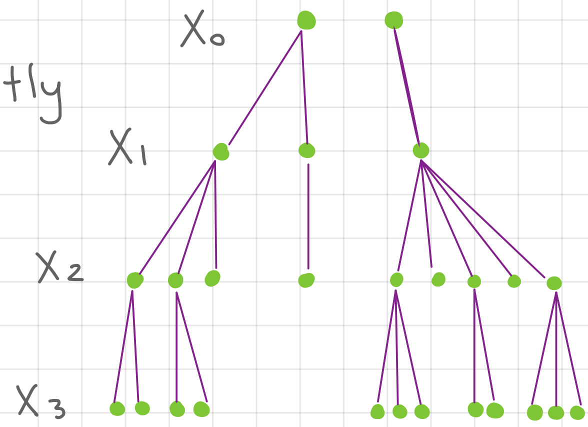
- Homework 4 is due on Friday, February 11, 11:59 PM

Galton-Watson Branching Process

Consider a population whose evolution (reproduction) is determined by the following rules

(i) Each individual produces k offsprings with probability p_k, where $\sum_{k=0}^{\infty} p_k = 1$

(ii) All individuals reproduce independently



Denote by X_n the size of the n-th generation. The number of individuals in the n-th generation depends only on the number of individuals in generation n-1 $\Rightarrow$ (X_n) is a Markov chain

If $X_n = 0$ for some n, then $\forall\, m > n$ $X_m = 0$ – extinction.

Galton-Watson Branching Process

Q: What is the probability that the population never goes extinct? $\mathbb{P}[X_n \geq 1 \;\; \forall n \in \mathbb{N}] = ?$

Direct computation: Let $\{Y_i\}_{i=1}^{\infty}$ be i.i.d. random variables with $Y_i \in \{0, 1, \ldots\}$ and $\mathbb{P}[Y_i = k] = p_k$. Then

$$p(i,j) = \mathbb{P}[X_{n+1} = j \mid X_n = i] = \mathbb{P}[Y_1 + \cdots + Y_i = j]$$

Distribution of $Y_1 + \cdots + Y_i$ is given by the i-fold convolution, which is hard to work with.

Instead, we study one particular quantity

$$q(i) = \mathbb{P}_i[X_n = 0 \text{ for some } n]$$

and develop a method to establish if $q(i) = 1$ or $q(i) < 1$

Galton-Watson Branching Process

Denote by $\mu := \sum_{k=0}^{\infty} k\, p_k$ the expected number of offsprings for each individual, $\mu = \mathbb{E}[Y_1]$

Then

$$\mathbb{E}_i[X_n] = \sum_{k=0}^{\infty} \mathbb{E}_i[X_n \mid X_{n-1} = k]\, \mathbb{P}_i[X_{n-1} = k]$$

and by the Markov property

$$\mathbb{E}_i[X_n \mid X_{n-1} = k] = \mathbb{E}[Y_1 + \cdots + Y_k] = k \cdot \mathbb{E}[Y_1] = k \cdot \mu$$

Thus

$$\mathbb{E}_i[X_n] = \sum_{k=0}^{\infty} k \cdot \mu \cdot \mathbb{P}_i[X_{n-1} = k] = \mu \cdot \mathbb{E}_i[X_{n-1}]$$

$$= \mu^n \mathbb{E}_i[X_0] = i \cdot \mu^n$$

But $\mathbb{E}_i[X_n] = \sum_{k=0}^{\infty} k \cdot \mathbb{P}_i[X_n = k] \geq \sum_{k=1}^{\infty} 1 \cdot \mathbb{P}_i[X_n = k] = \mathbb{P}_i[X_n \geq 1]$

Galton - Watson Branching Process

(1) If $\mu < 1$, then for any i the population gets extinct almost surely

$$q(i) = \mathbb{P}[X_n = 0 \text{ for some } n] = 1$$

- $\mathbb{P}_i[X_n \geq 1] \leq i\mu^n \Rightarrow \lim_{n\to\infty} \mathbb{P}_i[X_n \geq 1] = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}_i[X_n = 0] = 1$
- $\mathbb{P}_i[X_n = 0 \text{ for some } n] = \mathbb{P}_i\left[\bigcup_{n=0}^{\infty} \{X_n = 0\}\right]$

$$(X_n = 0 \Rightarrow X_{n+1} = 0) \overset{\text{monotonicity}}{\Rightarrow}$$

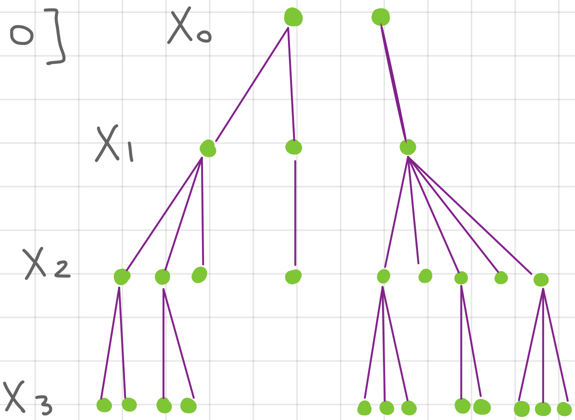
$$\mathbb{P}_i\left[\bigcup_{n=0}^{\infty} \{X_n = 0\}\right] = \lim_{n\to\infty} \mathbb{P}_i[X_n = 0] = 1$$

Def 15.1 Let $(X_n)_{n\geq 0}$ be a branching process with offspring distribution $p_0, p_1, \ldots$ and mean μ. We call (X_n) subcritical if $\mu < 1$; critical if $\mu = 1$; supercritical if $\mu > 1$.

Galton - Watson Branching Process

Subcritical GW branching process gets extinct with probability 1. What about the super-/critical regime?

Observation: denote $q_n(i) = \mathbb{P}_i[X_n = 0]$



(2) $q_n(i) = [q_n(1)]^i$

$X_n = 0$ iff for each of i independent subprocesses $X_n^{(j)} = 0$

We saw that

$$q(i) := \mathbb{P}_i[\exists n : X_n = 0] = \lim_{n\to\infty} q_n(i)$$ therefore

$q(i) = (q(1))^i$ and it is enough to compute $q(1) =: q$.

Q: How to compute $q = \mathbb{P}_1[\exists n : X_n = 0]$?

Probability generating function

Def Let Y be a random variable with values in $\{0,1,2,\dots\}$.
We call the function

$$\varphi_Y(s) := \mathbb{E}[s^Y] = \sum_{k=0}^{\infty} s^k \, \mathbb{P}[Y=k]$$

The probability generating function of Y.

Properties:

(1) $\varphi_Y(s)$ is analytic on $(-1,1)$; $\varphi_Y^{(n)}(0) = n! \, \mathbb{P}[Y=k]$

(2) $\varphi_Y(1) = 1$; $\varphi_Y(0) = \mathbb{P}[Y=0]$

(3) For $|s|<1$, $\varphi_Y'(s) = \sum_{k=1}^{\infty} k s^{k-1} \, \mathbb{P}[Y=k]$; if $\mathbb{E}[Y]<\infty$, then $\varphi_Y'(1) = \mathbb{E}[Y]$

(4) For $|s|<1$, $\varphi_Y''(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} \, \mathbb{P}[Y=k]$; in particular, if $\mathbb{P}[Y\geq 2]>0$, then $\varphi_Y(s)$ is (strictly) convex on $(0,1)$

Galton-Watson Branching Process

Theorem 15.2 Let $(X_n)_{n\geq 0}$ be a branching process with offspring distribution $p_0, p_1, \ldots$. Let φ be the probability generating function of this distribution $\varphi(s) = \sum_{k=0}^{\infty} p_k s^k$.
Then the extinction probability q is given by

$$q = \min\{ s \in [0,1] : \varphi(s) = s \}$$

Proof.

(i) $q(i) = q^i$

(ii) $q = \varphi(q)$

Using first step analysis

$$q = \mathbb{P}_1[\exists n : X_n = 0] = \sum_{i=0}^{\infty} \mathbb{P}_1[\exists n : X_n = 0 \mid X_1 = i]\, \mathbb{P}_1[X_1 = i]$$

$$= \sum_{i=0}^{\infty} \mathbb{P}_i[\exists n : X_n = 0]\, p_i = \sum_{i=0}^{\infty} q^i p_k = \varphi(q)$$

Galton - Watson Branching Process

(iii) $q \in [0,1]$, $\varphi(1) = 1$

(iv) Let $\hat{q} = \min\{s \in [0,1] : \varphi(s) = s\}$. Then $\forall n$ $\underset{\mathbb{P}_1[X_n=0]}{\overset{\parallel}{q_n}} \leq \hat{q}$

Induction: $q_0 = 0 \leq \hat{q}$

Suppose $q_{n-1} \leq \hat{q}$. Then

$$q_n = \mathbb{P}_1[X_n = 0] = \sum_{i=0}^{\infty} \mathbb{P}_1[X_n = 0 \mid X_1 = i]\, \mathbb{P}[X_1 = i]$$

$$= \sum_{i=0}^{\infty} \mathbb{P}_i[X_{n-1} = 0]\, p_i = \sum_{i=0}^{\infty} q_{n-1}^{i} p_i$$

Thus $q_n \leq \sum_{i=0}^{\infty} (\hat{q})^i p_i = \varphi(\hat{q}) = \hat{q}$. By the principle of mathematical induction $q_n \leq \hat{q}$ for all $n \in \mathbb{N}$

(v) $q = \varphi(q)$ and $q \in [0,1] \Rightarrow q \geq \hat{q}$

(vi) $\forall n$ $q_n \leq \hat{q} \Rightarrow \lim_{n \to \infty} q_n = q \leq \hat{q}$

$\Rightarrow q = \hat{q}$ ∎

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Q: When does $q<1$? When does $s=\varphi(s)$ for $s\in[0,1)$?

Remark If $p_1=1$, then $\mathbb{P}_1[X_n=1]=1$.

Corollary 15.3 Suppose $p_1\neq 1$. Then $q=1$ if the process is critical or subcritical, and $q<1$ if the process is supercritical.

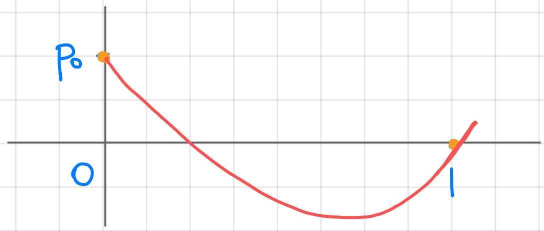
Proof. Subcritical: discussed before.

Supercritical: $\mu>1$. Denote $f(s)=\varphi(s)-s$. Then

- $f'(1)=\varphi'(1)-1=\mu-1>0$
- $f(0)=p_0$, $f(1)=\varphi(1)-1=0$
- $\exists\ s'\in(0,1)$ s.t. $f(s')<0$
- f is continuous on $[0,1]$

$\Rightarrow \exists\ s\in(0,s')$ s.t. $f(s)=0$

$\varphi(s)-s=0$



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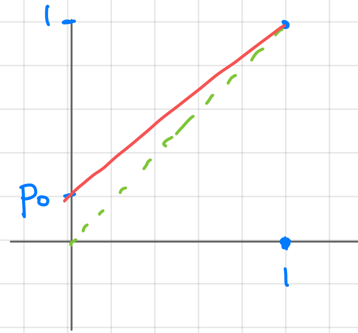

Critical: $\mu = 1$

- $\mu = 1 \Rightarrow p_0 \neq 1$ (otherwise $\mu = \sum_{k=0}^{\infty} k p_k = 0$)
- $\mu = 1 \Rightarrow p_0 \neq 0$ (otherwise $\sum_{k=1}^{\infty} k p_k = 1 \Rightarrow p_1 = 1$)
- $p_1 \neq 1 \Rightarrow \sum_{k=2}^{\infty} p_k > 0$ (otherwise $\mu = \sum_{k=0}^{\infty} k p_k = 0 \cdot p_0 + 1 \cdot p_1 = p_1 < 1$)
- $\varphi'(1) = 1$
- if $t \in (0,1)$, then $\varphi'(t) < 1$

$$\varphi'(t) = \sum_{k=1}^{\infty} k t^{k-1} p_k = p_1 + \sum_{k=2}^{\infty} k t^{k-1} p_k < p_1 + \sum_{k=2}^{\infty} k p_k = 1$$

- Take any $s \in (0,1)$, $\int_s^1 \varphi'(t)\,dt = \varphi(1) - \varphi(s) = 1 - \varphi(s) < \int_s^1 dt = 1 - s$

$\Rightarrow \varphi(s) > s$ for all $s \in (0,1)$