MATH 285: Stochastic Processes

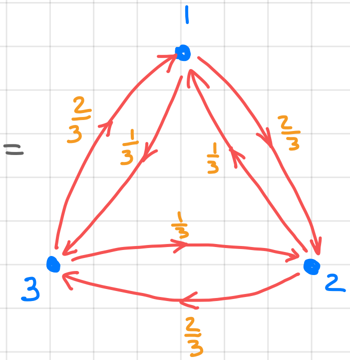
math-old.ucsd.edu/~ynemish/teaching/285

Today: MCMC

- Homework 3 is due on Friday, February 4, 11:59 PM

Example

Consider random walk on $G =$



Transition matrix

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix},$$

P is doubly stochastic, so

$\pi =$

Detailed balance equation:

$p(j,i) =$ $\Rightarrow$

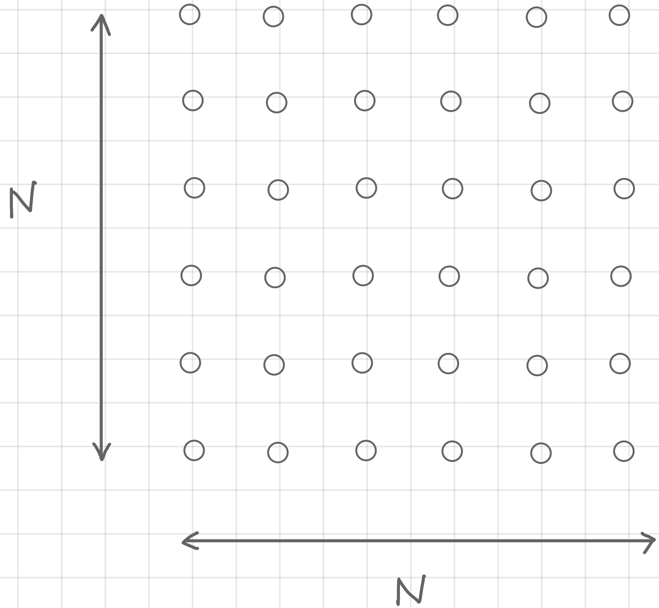
If $\pi = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|}\right)$, (X_n) is reversible only if

Example: Hard Core Configuration

Hard Core Configuration

on $\{1,2,\dots,N\}^2$ is a function

such that



Denote by HCC_N the set of all hard core configurations on $\{1,\dots,N\}^2$. Suppose we want to choose a uniform distribution on HCC_N, $\mathbb{P}[Z=c] = \qquad \forall c \in HCC_N$

Problem: How to compute $|HCC_N|$?

Example: Hard Core Configuration

Computing $|HCC_N|$ for large N is difficult.

Instead we construct a MC on HCC_N whose stationary distribution is the uniform distribution on HCC_N.

Construction: for any two configurations $c \neq c'$

$$p(c, c') = \begin{cases} \quad\quad , & \text{if } c \text{ and } c' \text{ differ at exactly one point} \\ \quad\quad , & \text{otherwise} \end{cases}$$

$$p(c, c) =$$

Implementation: at each step choose $(i,j) \in \{1, \dots, N\}^2$ uniformly at random and change the value at (i,j) if possible. E.g., $X_n = c$, choose (i,j).

- If $c(i,j) = 1$, then

Example : Hard Core Configuration

- If $c(i,j)=0$, and $c(i\pm1, j\pm1)=0$, then

- If $c(i,j)=0$ and one of $c(i\pm1, j\pm1) \neq 0$, then

(i) Then for any $c, c' \in HCC_N$ $\mathbb{P}[X_{n+1}=c' \mid X_n=c]=$

(ii) (X_n) is irreducible ($\forall$ $c, c' \in HCC_N$

(iii) $p(c,c') = p(c',c)$ (c and c' differ in only one coordinate)

(iv) Uniform distribution on HCC_N is the stationary distribution

$\pi(c) = 1/|HCC|_N$ $\Rightarrow$

$\Rightarrow$

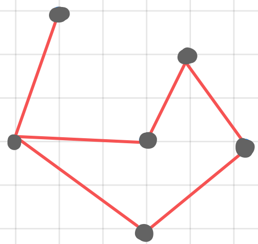
Now if we start the process from any $c \in HCC_N$, then

for sufficiently large n $\mathbb{P}[X_n = c] \approx$

Example: Graph coloring

Let $G=(V,E)$ be a finite graph. A q-coloring of G (with $q\in\mathbb{N}$) is a function $f: V\to\{1,2,\dots,q\}$ s.t.

(different colors of neighboring vertices)



Q: How to choose a q-coloring uniformly at random?

Construct a MC: if f and g are two q-colorings, $f\neq g$,

set $p(f,g)=\begin{cases} & \text{if } f \text{ and } g \text{ differ at exactly one vertex} \\ & \text{otherwise}\end{cases}$

$p(f,f)=$

(X_n) with transition probabilities $p(f,g)$ is an irreducible MC with stationary distribution $\pi(f)=$

Metropolis - Hastings Algorithm

Q: How to sample any (strictly positive) distribution π ?

Two-step MC: (1) propose moves (2) accept/reject move

Construction of the Markov Chain

Let S be a finite set, $\pi > 0$ a distribution on S.

(1) Construct an irreducible MC on S with symmetric transition probabilities

(2) If π (the desired distribution) is not uniform, construct a new MC with transition probabilities

$$p(i,j) =$$

- $\pi(i)\, p(i,j) =$

so π is stationary for $p(i,j)$

Metropolis - Hastings Algorithm

Suppose we know how to simulate a MC with transition probabilities $q(i,j)$. The we can simulate a MC with transition probabilities $p(i,j)$ using the two - step algorithm:

(i) Propose the move:

If $X_n = i$, then

(ii) Accept or reject the move:

Accept the move with probability

We get that $\mathbb{P}[X_{n+1} = j \mid X_n = i] =$

If we now run (X_n) sufficiently long, then

Q: How long should we run (X_n)?

Convergence rate

Suppose that (X_n) is irreducible and aperiodic MC on S with $|S| = N$, and suppose that P is symmetric, $P = P^t$.

Then by the spectral theorem

Perron - Frobenius theorem $\Rightarrow$ $\lim_{n \to \infty} P^n =$

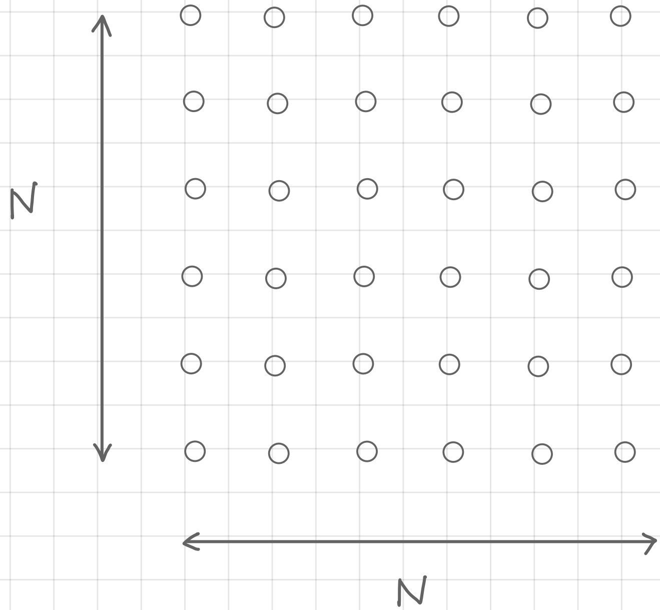
Then $\| P^n - P^\infty \| =$. Mixing time: n s.t.

E.g. for $\begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$

for $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Example: Ising model

- $\Lambda_N =$
- Spin configuration:



- Energy: $H(\sigma) =$
- Gibbs measure: $P_\beta(\sigma) =$

  where $Z_\beta =$ is the partition function (difficult)
- Take . Then
- For $\sigma \neq \sigma'$ take $q(\sigma, \sigma') = \{$
- Run MC (X_n) with $p(\sigma, \sigma') =$