MATH 285: Stochastic Processes

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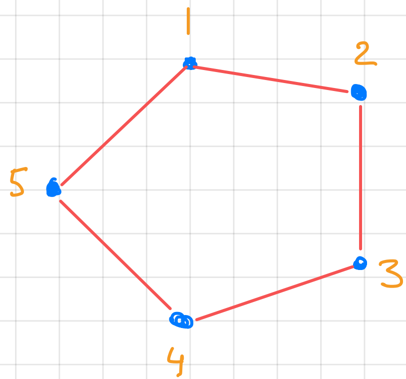
Today: Time reversal

- Homework 3 is due on Friday, February 4, 11:59 PM

Stationary distribution

Example (X_n) SSRW on $G =$



- (X_n) is irreducible
- (X_n) is aperiodic
- P is doubly stochastic i.e. $\sum_{i \in S} p(i,j) = 1 \quad \forall j \in S$

  Remark: if P is doubly stochastic with finite state space S, then $\pi = \left(\frac{1}{|S|}, \dots, \frac{1}{|S|} \right)$

- $\pi = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right)$

- $\forall i, \quad \mathbb{E}_i[T_i] = \frac{1}{\pi(i)} = 5$

- $\forall i,j \quad \gamma(i,j) = \mathbb{E}_i[T_i] \cdot \pi(j) = 1$

Time reversal

Theorem 13.2 Let (X_n) be an irreducible Markov chain possessing a stationary distribution π. Let $N \in \mathbb{N}$, $\mathbb{P}[X_0 = j] = \pi(j)$
and for $0 \le n \le N$ define $Y_n = X_{N-n}$. Then $(Y_n)_{0 \le n \le N}$ is an irreducible Markov chain with the same stationary distribution, and transition probabilities $q(i,j)$ given by

$$\pi(j) q(j,i) = \pi(i) p(i,j) \quad \forall\ i,j$$

Proof. (i) By Corollary 10.2 (or 11.1) $\pi(j) > 0 \quad \forall j$

(ii) $\sum_{i \in S} q(j,i) = 1$

$$\sum_{i \in S} q(j,i) = \sum_{i \in S} \frac{\pi(i)}{\pi(j)} p(i,j) = \frac{1}{\pi(j)} \cdot \pi(j) = 1$$

Time reversal

(iii) $\sum_{j\in S} \pi(j)\, q(j,i) = \pi(i)$

$$\sum_{j\in S} \pi(j)\, q(j,i) = \sum_{j\in S} \pi(i)\, p(i,j) = \pi(i)\cdot 1 = \pi(i)$$

(iv) $(Y_n)_{0\le n\le N}$ is Markov with initial distribution π and transition probabilities $q(i,j)$

- Enough to show that for any sample path $(i_0, i_1, \dots, i_N)$

$$\mathbb{P}[Y_0 = i_0, Y_1 = i_1, \dots, Y_N = i_N] = \pi(i_0)\, q(i_0,i_1) \cdots q(i_{N-1}, i_N)$$

- $\mathbb{P}[Y_0 = i_0, Y_1 = i_1, \dots, Y_N = i_N] = \mathbb{P}[X_0 = i_N, X_1 = i_{N-1}, \dots, X_N = i_0]$

$$= \pi(i_N)\, p(i_N, i_{N-1}) \cdots p(i_1, i_0) = \pi(i_{N-1})\, q(i_{N-1}, i_N)\, p(i_{N-1}, i_{N-2}) \cdots$$

$$= \pi(i_{N-1})\, p(i_{N-1}, i_{N-2}) \cdots p(i_1, i_0)\, q(i_{N-1}, i_N) = \cdots =$$

$$= \pi(i_1)\, p(i_1, i_0)\, q(i_1, i_2) \cdots q(i_{N-1}, i_N) = \pi(i_0)\, q(i_0, i_1) \cdots q(i_{N-1}, i_N)$$

Time reversal

$$\pi(j)\, q(j,i) = \pi(i)\, p(i,j)$$

(v) (Y_n) is irreducible

Take any $i, j \in S$.

(X_n) is irreducible $\Rightarrow$ there exists $n \in \mathbb{N}$ and $i_1, \dots, i_n \in S$

s.t. $\quad p(i,i_1) \cdot p(i_1,i_2) \cdot \dots \cdot p(i_n,j) > 0$

$$\Rightarrow \quad q_n(j,i) \geq q(j,i_n) \cdot q(i_n, i_{n-1}) \cdots q(i_1,i)$$

$$= \frac{\pi(i)}{\cancel{\pi(i_1)}} p(i,i_1) \frac{\cancel{\pi(i_1)}}{\cancel{\pi(i_2)}} p(i_1,i_2) \cdots \frac{\cancel{\pi(i_n)}}{\pi(j)} p(i_n,j)$$

$$= \frac{\pi(i)}{\pi(j)} p(i,i_1) \cdots p(i_n,j) > 0$$

$\Rightarrow (Y_n)$ is irreducible $\blacksquare$

The chain $(Y_n)_{0 \leq n \leq N}$ is called the time-reversal of $(X_n)_{0 \leq n \leq N}$

Time reversibility

Q: When does the time-reversal have the same transition probabilities?

<u>Def 13.5</u> Let (X_n) be an irreducible MC with state space S (finite or countable), initial distribution λ and transition probabilities $p(i,j)$. We call (X_n) **reversible** if, for all $N>1$, $(X_{N-n})_{0\le n\le N}$ is also an irreducible MC with init. distr. λ and trans. prob. $p(i,j)$.

<u>Def 13.10</u> Let (X_n) be a MC with initial distribution λ and transition probabilities $p(i,j)$. We say that λ and $p(i,j)$ are in **detailed balance** (satisfy the detailed balance equation) if for all i,j $\quad \lambda(i)\,p(i,j) = \lambda(j)\,p(j,i)$

Time reversibility

Thm 13.11 If the initial distribution λ and the transition probabilities $p(i,j)$ are in detailed balance, then λ is the stationary distribution for $p(i,j)$

Proof

$$\sum_{i \in S} \lambda(i) p(i,j) = \sum_{i \in S} \lambda(j) p(j,i) = \lambda(j) \qquad \blacksquare$$

Thm 13.12 Let (X_n) be an irreducible MC with initial distribution λ and transition probabilities $p(i,j)$. Then (X_n) is reversible iff λ and $p(i,j)$ are in detailed balance

Proof $(\Rightarrow)$ (X_n) reversible $\Rightarrow \mathbb{P}[X_N = j] = \lambda(j) \; \forall N \in \mathbb{N}, \forall j \in S$

$\Rightarrow \lambda$ is stationary $\overset{T13.2}{\Longrightarrow} \forall i,j \quad \lambda(i) p(i,j) = \lambda(j) p(j,i) \; \forall i,j$

$(\Leftarrow)$ By Thm 13.11 λ is stationary $\overset{T13.2}{\Longrightarrow} q(j,i) = \frac{\lambda(i)}{\lambda(j)} p(i,j) = p(j,i) \qquad \blacksquare$

Detailed balance

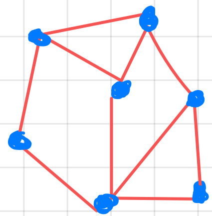
If (X_n) is irreducible and reversible, then (X_n) possesses a stationary distribution π and

$$\pi(j)\,p(j,i) = \pi(i)\,p(i,j).$$

It is usually easier to solve the detailed balance equation than $\pi = \pi P$.

Example Let G be a finite graph with no isolated vertices. Let (X_n) be a SSRW on G,

$p(i,j) = \frac{1}{v_i}$, $i \sim j$, where $v_i = \#\{j : i \sim j\}$, valency



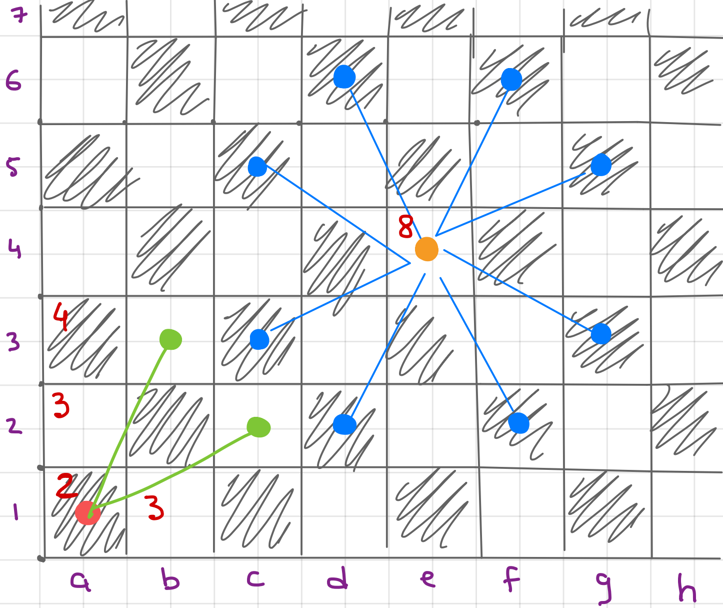
Detailed balance: $\pi(i)\,p(i,j) = \pi(j)\,p(j,i)$

Notice that $v_i\,p(i,j) = \begin{cases} 1, & i \sim j \\ 0, & i \not\sim j \end{cases}$, so $v_i\,p(i,j) = v_j\,p(j,i)$

Thus $\pi(i) := v_i \big/ \sum_{j \in V} v_j$ satisfies the detailed balance equation.

Example

Consider a chessboard (8×8) and a random knight that makes each permissible move with equal probability. Suppose that the knight starts in one of the corners.



How long on average will it take to return?

Consider the graph with $V = \{1, \dots, 8\}^2$ and $i \sim j$ if the knight can go directly from i to j. The the knight performs a SSRW on G. To find the stationary distribution count the valencies: $\sum_i v_i = 336$, $\pi(a1) = \frac{2}{336} = \frac{1}{168}$, $\mathbb{E}_{a_1}[T_{a_1}] = 168$