MATH 285: Stochastic Processes

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Today: Positive and null recurrence

- Homework 2 is due on Friday, January 21 11:59 PM

Positive recurrence and stationary distribution

Def 9.2 Let i be a recurrent state for MC (X_n).
Denote $T_i = \min\{n \geq 1 : X_n = i\}$. If $\mathbb{E}_i[T_i] < \infty$, then we call i positive recurrent. If $\mathbb{E}_i[T_i] = \infty$, the we call i null recurrent.

Prop 9.4 In a finite-state irreducible Markov chain all states are positive recurrent.

Thm 10.2 Let (X_n) be a time homogeneous MC with state space S, and suppose that the chain possesses a stationary distribution π.

(1) If (X_n) is irreducible, then $\pi(j) > 0$ for all $j \in S$

(2) In general, if $\pi(j) > 0$, then j is positive recurrent.

Positive recurrence and stationary distributions

Thm 9.6 Let (X_n) be a Markov chain with a state space that is countable (but not necessarily finite).

Suppose there exists a positive recurrent state $i \in S$, $\mathbb{E}_i[T_i] < \infty$.

For each state $j \in S$ define

$$\gamma(i,j) = \mathbb{E}_i\left[\sum_{n=0}^{T_i - 1} \mathbb{1}_{\{X_n = j\}}\right]$$

(the expected number of visits to j before reaching i).

Then the function $\pi : S \to [0,1]$

$$\pi(j) = \frac{\gamma(i,j)}{\mathbb{E}_i[T_i]}$$

is a stationary distribution for (X_n).

Positive recurrence and stationary distributions

Proof of Thm. 9.6 Recall $T_i = \min\{n \geq 1 : X_n = i\}$.

(i) $\sum_{j \in S} \gamma(i,j) =$

$\sum_{j \in S} \gamma(i,j) =$

(ii) Enough to show that $\forall j$

Denote $\tilde{\pi} = (\tilde{\pi}(j))_{j \in S}$ with $\tilde{\pi}(j) := \frac{\gamma(i,j)}{\mathbb{E}_i[T_i]}$. Then $\forall j \in S$

- $\tilde{\pi}(j) \geq 0$
- $\tilde{\pi}(j) = \sum_{k \in S} \tilde{\pi}(k)\, p(k,j)$
- $\sum_{j \in S} \tilde{\pi}(j) = \sum_{j \in S} \frac{\gamma(i,j)}{\mathbb{E}_i[T_i]} = 1$

(iii) $\forall j \quad \gamma(i,j) = \sum_{k \in S} \gamma(k)\, p(k,j)$

- Given that $X_0 = 0$, for any $j \in S$

Positive recurrence and stationary distributions

- $\gamma(i,j) = \mathbb{E}_i\left[\sum_{n=1}^{T_i} \mathbb{1}_{\{X_n = j\}} \right] =$

- For any $n \geq 1$ and $j \in S$

$$\mathbb{P}_i[\, n \leq T_i,\ X_n = j \,] = \sum_{k \in S} \mathbb{P}_i[\, n \leq T_i,\ X_n = j,\ X_{n-1} = k \,]$$

- $\gamma(i,j) = \sum_{k \in S} p(k,j) \sum_{n=1}^{\infty} \mathbb{P}_i[\, X_{n-1} = k,\ n \leq T_i \,]$

Positive recurrence and stationary distributions

Corollary 10.1 If i is a positive recurrent state, then the stationary distribution π defined in Thm 9.6 satisfies

Proof. Follow from Thm 9.6 and $\gamma(i,i) = 1$. ∎

Corollary 11.1 For an irreducible Markov chain, TFAE

(1) there exists a stationary distribution with all entries > 0

(2) there exists a stationary distribution

(3) there exists a positive recurrent state

(4) all states are positive recurrent

Proof.

Example: Birth and death chain

Let (X_n) be a birth and death chain: $S = \{0, 1, \dots\}$

- $p(i, i+1) = q, \quad p(i, i-1) = 1-q \quad$ for $\quad i \geq 1$
- $p(0,1) = q, \quad p(0,0) = 1-q$

$q \in (0,1)$

(X_n) is irreducible

Q: Does stationary distribution exist?

$$\begin{cases} \pi(0) = \pi(0)(1-q) + \pi(1)(1-q) \\ \pi(i) = \pi(i-1)\,q + \pi(i+1)(1-q), \quad i \geq 1 \end{cases} \qquad \Big\{$$

$$\begin{cases} \pi(0) - \pi(1) = \\ \pi(i) - \pi(i+1) = \end{cases}$$

$\sum_{i=0}^{\infty} \pi(i) = 1$, so stationary distribution exists

$\Longleftrightarrow \quad \sum_{i=0}^{\infty} \beta^i \pi(0) = 1 \quad \Longleftrightarrow \qquad \Longleftrightarrow$

Example: Birth and death chain

$\beta < 1 \iff q < \frac{1}{2}$

- If $q < \frac{1}{2}$, then $\sum_{i=0}^{\infty} \beta^i =$, $\begin{cases} \pi(0) = \\ \pi(i) = \end{cases}$

  All states are positive recurrent.

- If $q = \frac{1}{2}$, then (X_n) is not positive recurrent.

  (X_n) is recurrent: if $(\tilde{X}_n)$ is a SSRW on $\mathbb{Z}$, then

  $\mathbb{P}_0[\tilde{T}_0 < \infty] = 1 =$

  $=$

  At the same time $\mathbb{P}_0[T_0 < \infty] =$

  and $\mathbb{P}_0[T_0 < \infty \mid X_0 = 1] =$

  We conclude that (X_n) is null recurrent

- If $q > \frac{1}{2}$, then (X_n) is transient

Ergodic Theorem

<u>Thm 11.3</u> Let (X_n) be an irreducible recurrent Markov chain with state space S. Let $j \in S$. Define

$(\pi(j) = 0$ if $\mathbb{E}_j[T_j] = \infty)$.

Let $V_n(j) := \sum_{m=1}^{n} \mathbb{1}_{\{X_n = j\}}$ be the number of visits to state j up to time n. Then for any state $i \in S$

and

<u>Proof.</u> (i) $\mathbb{P}_i[\, V_n(j) \to \infty$ as $n \to \infty \,] = 1$

Otherwise $\mathbb{P}_j[\, (X_n)$ visits j finitely many times $] > 0$

Ergodic Theorem

Denote by T_j^k the time of the k-th visit to state j.

(ii) $T_j^{V_n(j)} \leq n \leq T_j^{V_n(j)+1}$

(iii)

Repeating the proof from Thm 10.2 we have that

$$\mathbb{P}_i\left[\frac{T_j^k}{k} \to \mathbb{E}_j[T_j] \text{ as } k \to \infty \right] = 1$$

By (i) $\lim_{n\to\infty} \frac{T_j^{V_n(j)}}{V_n(j)} = \lim_{k\to\infty} \frac{T_j^k}{k}$

(iv) By the Squeeze lemma , therefore

. By definition $\pi(j) = \frac{1}{\mathbb{E}_j[T_j]}$.

Ergodic Theorem

(v) $\frac{1}{n}\sum_{m=1}^{n} p_m(i,j) =$

(vi) $\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_i[V_n(j)] =$