MATH 285: Stochastic Processes

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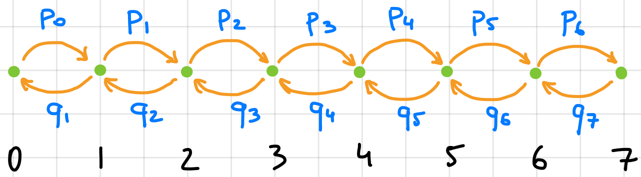
Today: Positive and null recurrence

- Homework 2 is due on Friday, January 21 11:59 PM

Birth and death processes (infinite state space)



$S = \{0, 1, 2, 3, \dots\}$

$p(i, i+1) = p_i, \quad p(i, i-1) = 1 - p_i \overset{q_i}{=}$

$p(0,1) = p_0, \quad p(0,0) = 1 - p_0$

$p_0 \in [0,1], \quad p_0 = 0$ absorbing, $p_0 = 1$ reflecting

Model of population growth : X_n = size of the population at time n

$\mathbb{P}_i[\exists n \geq 0 : X_n = 0]$ – extinction probability

$\mathbb{P}_i[X_n \to \infty \text{ as } n \to \infty]$ – probability that population explodes

Denote $h(i) := \mathbb{P}_i[\exists n \geq 0 \ X_n = 0] = \mathbb{P}_i[\tau_0 < \infty]$, $\tau_0 = \min\{n \geq 0 : X_n = 0\}$

First step analysis:

<u>Theorem 7.0</u> $(h(0), h(1), \dots)$ is the minimal solution to
$$\begin{cases} h(0) = 1 \\ h(i) = \sum_{j=0}^{\infty} p(i,j) h(j) \end{cases}$$

Positive and null recurrence

Let (X_n) be a Markov chain, and let i be a recurrent state. Starting from i, (X_n) revisits i infinitely many times, $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1$

How often does (X_n) revisit state i?

(i) After n steps, (X_n) revisits i $\approx \frac{n}{2}$ times, spends half of the time at i

(ii) After n steps, (X_n) revisits i $\approx \sqrt{n}$ times, the fraction of time spent at i tend to 0 as $n \to \infty$, $\frac{\sqrt{n}}{n} \to 0$, $n \to \infty$

Def 9.2 Let i be a recurrent state for MC (X_n).
Denote $T_i = \min\{n \geq 1 : X_n = i\}$. If $\mathbb{E}_i[T_i] < \infty$, then we call i positive recurrent. If $\mathbb{E}_i[T_i] = \infty$, the we call i null recurrent.

Positive and null recurrence

Remark If i is recurrent, then $\mathbb{P}_i[T_i < \infty] = 1$. But it is still possible that $\mathbb{E}[T_i] = \infty$ or that $\mathbb{E}[T_i] < \infty$.

Example: $Y_1, Y_2 \in \mathbb{N}$, $\mathbb{P}[Y_1 = k] = \left(\frac{1}{2}\right)^k$, $Y_2 = 2^{Y_1}$, $\mathbb{P}[Y_2 = 2^k] = \left(\frac{1}{2}\right)^k$.

$$\mathbb{P}[Y_1 < \infty] = \mathbb{P}[Y_2 < \infty] = 1, \quad \mathbb{E}[Y_1] = 2, \quad \mathbb{E}[Y_2] = \infty$$

Prop 9.4 In a finite-state irreducible Markov chain all states are positive recurrent.

Proof. Fix state $j \in S$

(1) There exist $N \in \mathbb{N}$ and $q \in (0,1)$ such that for any $i \in S$

$\mathbb{P}_i[T_j \leq N] \geq q$ (probability of reaching j from i in the next N steps)

Since (X_n) is irreducible, $\forall i \ \exists \ n(i)$ s.t. $p_{n(i)}(i,j) > 0$

Take $N = \max\{n(i) : i \in S\}$, $q = \min\{p_{n(i)}(i,j) : i \in S\}$

Positive and null recurrence

(2) For any $i \in S$ $\quad \mathbb{P}_i[T_j > N] \leq 1-q < 1$. | follows from (1)

(3) For any $k \in \mathbb{N}$, $\quad \mathbb{P}_j[T_j > (k+1)N] \leq (1-q)^{k+1}$

For any $i \in S$ $\quad \mathbb{P}_j[T_j > (k+1)N \mid T_j > kN, X_{kN} = i] \overset{(SMP)}{=} \mathbb{P}_i[T_j > N] \leq 1-q$.

$$\mathbb{P}_j[T_j > (k+1)N] = \mathbb{P}_j[T_j > (k+1)N, T_j > kN]$$

$$= \sum_{i \in S} \mathbb{P}_j[T_j > (k+1)N, T_j > kN, X_{kN} = i]$$

$$= \sum_{i \in S} \mathbb{P}_j[T_j > (k+1)N \mid T_j > kN, X_{kN} = i] \, \mathbb{P}_j[T_j > kN, X_{kN} = i]$$

$$\leq (1-q) \, \mathbb{P}_j[T_j > kN]$$

Now repeat k times.

Positive and null recurrence

(4) $\mathbb{E}_j[T_j] = \sum_{n=1}^{\infty} \mathbb{P}_j[T_j \geq n] = \sum_{k=0}^{\infty} \sum_{n=kN+1}^{(k+1)N} \mathbb{P}_j[T_j \geq n]$

(5) $\mathbb{P}_j[T_j \geq n]$ is decreasing with n | $T_j \geq n \Rightarrow T_j \geq n-1$

Therefore $\forall\, n \in \{kN+1, \dots, (k+1)N\}$

$$\mathbb{P}_j[T_j \geq n] \leq \mathbb{P}_j[T_j \geq kN] \leq (1-q)^k$$

(6) $\sum_{n=kN+1}^{(k+1)N} \mathbb{P}_j[T_j \geq n] \leq N(1-q)^k$

Finally, $\mathbb{E}_j[T_j] \leq \sum_{k=0}^{\infty} N(1-q)^k = \frac{N}{q} < \infty$ ∎

Conclusion: All states of an irreducible MC with finite state space are positive recurrent.

Positive recurrence and stationary distributions

Thm 9.6 Let (X_n) be a Markov chain with a state space that is countable (but not necessarily finite).
Suppose there exists a positive recurrent state $i \in S$, $\mathbb{E}_i[T_i] < \infty$.
For each state $j \in S$ define

$$\gamma(i,j) = \mathbb{E}_i\left[\sum_{n=0}^{T_i-1} \mathbb{1}_{\{X_n=j\}}\right]$$

(the expected number of visits to j before reaching i).
Then the function $\pi : S \to [0,1]$

$$\pi(j) = \frac{\gamma(i,j)}{\mathbb{E}_i[T_i]}$$

is a stationary distribution for (X_n).

Proof. Next lecture.

Positive recurrence and stationary distribution

Thm 10.2 Let (X_n) be a time homogeneous MC with state space S, and suppose that the chain possesses a stationary distribution π.

(1) If (X_n) is irreducible, then $\pi(j)>0$ for all $j \in S$

(2) In general, if $\pi(j)>0$, then j is positive recurrent.

Proof. (1) Fix $j \in S$.

- π is stationary $\Rightarrow \pi = \pi P = \pi P^n \Leftrightarrow \pi(j) = \sum_{i \in S} \pi(i) p_n(i,j) \ \forall n$
- π is distribution $\Rightarrow \exists\, i_0 \in S$ s.t. $\pi(i_0)>0$
- (X_n) is irreducible $\Rightarrow \exists\, n_0 \in \mathbb{N}$ s.t. $p_{n_0}(i_0,j)>0$

$$\Rightarrow \pi(j) = \sum_{i \in S} \pi(i) p_{n_0}(i,j) \geq \pi(i_0) p_{n_0}(i_0,j) > 0$$

Positive recurrence and stationary distribution

(2) Suppose that $\pi(j) > 0$ and j is not positive recurrent.

(i) $\mathbb{E}_\pi \left[\sum_{m=1}^{n} \mathbb{1}_{\{X_m = j\}} \right] = n\,\pi(j)$

$\left(\mathbb{E}_\pi : \text{ initial distribution is } \pi,\ \mathbb{P}_\pi[X_0 = i] = \pi(i) \right)$

<u>Proof:</u> $\mathbb{E}_\pi \left[\sum_{m=1}^{n} \mathbb{1}_{\{X_m = j\}} \right] = \sum_{m=1}^{n} \mathbb{P}_\pi [X_m = j] = \sum_{m=1}^{n} \pi(j) = n\,\pi(j)$

Denote $V_n(j) := \sum_{m=1}^{n} \mathbb{1}_{\{X_m = j\}}$, $T_j^k = \min\{n \geq 0 : V_n(j) = k\}$ – time of k-th visit to j

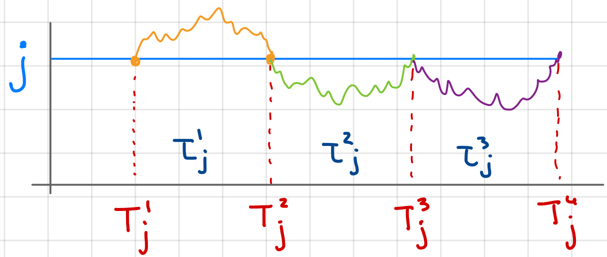
(ii) $\mathbb{P}_\pi \left[\lim_{k \to \infty} \frac{T_j^k}{k} = \infty \right] = 1$

<u>Proof:</u> If j is transient, then $\mathbb{P}_\pi [\exists k : T_j^k = \infty] = 1$ (visiting j only finitely many times).

Positive recurrence and stationary distribution

Suppose that j is null recurrent. Denote

$\tau_j^k := T_j^{k+1} - T_j^k$, so that $T_j^{k+1} = T_j^1 + \tau_j^1 + \tau_j^2 + \cdots + \tau_j^k$



- τ_j^k are stopping times
- SMP implies that

  $\{\tau_j^1, \tau_j^2, \dots\}$ are i.i.d.,
- τ_j^k have the same distribution as

  $T_j = \min\{n \geq 1 : X_n = j\}$ starting from j
- j is null recurrent $\Rightarrow$ $\mathbb{E}_j[T_j] = \infty$
- $$\frac{T_j^k}{k} = \frac{T_j^1 + \tau_j^1 + \cdots + \tau_j^{k-1}}{k} = \frac{T_j^1}{k} + \frac{\tau_j^1 + \cdots + \tau_j^{k-1}}{k-1} \cdot \frac{k-1}{k}$$
- $$\mathbb{P}_\pi\left[\lim_{k\to\infty} \frac{T_j^1}{k} = 0\right] = 1, \quad \mathbb{P}_\pi\left[\lim_{k\to\infty} \frac{\tau_j^1 + \cdots + \tau_j^{k-1}}{k-1} \cdot \frac{k-1}{k} = \infty\right] \overset{\text{SLLN}}{=} 1$$

Positive recurrence and stationary distribution

(iii) $\pi(j) = 0$

$$V_n(j) := \sum_{m=1}^{n} \mathbb{1}_{\{X_m = j\}}$$

Fix any $M > 0$.

- (ii) $\Rightarrow \exists N$ s.t. $\mathbb{P}_\pi\left[\frac{T_j^N}{N} \le M\right] \le \frac{1}{M}$
- $T_j^N / N \le M$ is equivalent to $\min\{n \ge 0 : V_n(j) = N\} \le MN$

$V_{MN}(j) \ge N$ implies $T_j^N \le MN$, therefore

$$\mathbb{P}_\pi[V_{MN}(j) \ge N] \le \mathbb{P}_\pi[T_j^N \le MN] \le \frac{1}{M}$$

- $\mathbb{E}_\pi[V_{MN}(j)] \overset{(i)}{=} MN\,\pi(j) = \sum_{k=1}^{MN} \mathbb{P}_\pi[V_{NM}(j) \ge k]$
- $\sum_{k=1}^{MN} \mathbb{P}_\pi[V_{MN}(j) \ge k] = \sum_{k=1}^{N} + \sum_{k=N+1}^{MN} \le N + (M-1)N \cdot \frac{1}{M} < 2N$
- $MN\pi(j) < 2N \Rightarrow \pi(j) < \frac{2}{M}$ (for all $M > 0$)

Conclusion: $\pi(j) = 0$, contradistion $\Rightarrow$ j is positive recurrent. ∎