Math 285: Winter 2022

Homework 8

Upload the homework to Gradescope by Wednesday, March 16, 11:59 PM.

1. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Let $X_t = B_t - tB_1$ for $0 \leq t \leq 1$. Then $(X_t)_{0\leq t\leq 1}$ is called a *Brownian bridge*. Note that $X_0 = X_1 = 0$.

(a) As shown in lecture, $(X_t)_{t\geq 0}$ is a Gaussian process. For $0 \leq s \leq t \leq 1$, calculate $\mathbb{E}[X_t]$ and $\mathrm{Cov}(X_s, X_t)$.

(b) For $0 \leq t \leq 1$, let $Y_t = X_{1-t}$. Show that $(Y_t)_{0\leq t\leq 1}$ is also a Brownian bridge.

(c) Show that, if $(X_t)_{0\leq t\leq 1}$ is a Brownian bridge and $W_t = (t+1)X_{t/(t+1)}$ for all $t \geq 0$, then $(W_t)_{t\geq 0}$ is a standard Brownian motion.

2. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For all $t \in \mathbb{R}$, let $Z_t = e^{-t}B_{e^{2t}}$. The process $(Z_t)_{t\in\mathbb{R}}$ is called the *Ornstein-Uhlenbeck process*.

(a) As shown in class, $(Z_t)_{t\in\mathbb{R}}$ is a Gaussian process. Show that, in fact, Z_t is a *standard* normal random variable for each $t \in \mathbb{R}$.

(b) Calculate $\mathrm{Cov}(Z_s, Z_t)$ for any $s, t \in \mathbb{R}$.

(c) Show that the Ornstein-Uhlenbeck process is *stationary*: the process $(Z_{s+t})_{t\in\mathbb{R}}$ has the same marginal distributions as $(Z_t)_{t\in\mathbb{R}}$ for any $s \in \mathbb{R}$.

3. Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be independent standard Brownian motions. Let $C_t = A_t - B_t$ for $t \geq 0$.

(a) Show that $(C_t)_{t\geq 0}$ is a Brownian motion, and find its variance parameter.

(b) Show that, with probability 1, $A_t = B_t$ for infinitely many $t \geq 0$.

4. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Compute the following probabilities exactly, or to 3 decimal places.

(a) $\mathbb{P}(B_2 > B_1)$.

(b) $\mathbb{P}(B_3 > B_2 > B_1 + 2)$.

(c) $\mathbb{P}(B_t \leq 1 \text{ for all } t \leq 4)$.

(d) $\mathbb{P}(B_t < 0 \text{ for all } t > 20)$.