Math 285: Winter 2022

Homework 4

Due: Friday, February 11, 11:59 PM

Upload the homework to Gradescope by Friday, February 11, 11:59 PM. Late homework will not be accepted.

1. (a) Show that every irreducible 2-state Markov chain is reversible.

(b) Give an example of an irreducible 3-state Markov chain that is not reversible.

2. Let $p(\cdot, \cdot)$ be the transition kernel for an irreducible, reversible Markov chain with state space S. For any states $j_1, \dots, j_n$, show that the following *cycle condition* is satisfied:

$$p(j_1, j_2)p(j_2, j_3) \cdots p(j_{n-1}, j_n)p(j_n, j_1) = p(j_1, j_n)p(j_n, j_{n-1}) \cdots p(j_3, j_2)p(j_2, j_1).$$

3. Let $\alpha > 1$. Suppose $(X_n)_{n \geq 0}$ is a Markov chain with state space $\mathbb{N}$, and transition probabilities $p(i, i-1) = 1$ for $i \geq 1$, $p(0,0) = 0$, and $p(0, j) = Cj^{-\alpha}$ where $1/C = \sum_{j=1}^{\infty} j^{-\alpha}$.

(a) Show that $(X_n)_{n \geq 0}$ is irreducible, recurrent, and non-reversible for all α.

(b) For which values of α is the chain positive recurrent, and for which values of α is the chain null recurrent?

4. Consider a Galton-Watson branching process started with a single individual in generation 0. Assume each individual has two offspring in the next generation with probability p and zero offspring in the next generation with probability $1 - p$.

(a) For what values of p is this process subcritical, critical, and supercritical?

(b) Find the probability that the population eventually goes extinct, as a function of p.