Math 285: Winter 2022
Homework 2

Due: Friday, January 28, 11:59 PM

Upload the homework to Gradescope by Friday, January 28, 11:59 PM. Late homework will not be accepted.

1. The rooted binary tree is an infinite graph T with one distinguished vertex r (the root) from which comes a single edge; at every other vertex there are three edges (and no loops). Show that the simple random walk on T is transient.

2. Fix $\alpha \in (0, \infty)$. Let $(X_n^{(\alpha)})_{n=1}^{\infty}$ be a Markov chain with state space $\mathbb{N} = \{0, 1, 2, \ldots\}$ and transition probabilities $p(i, j) = 0$ unless $|i - j| = 1$, and satisfying

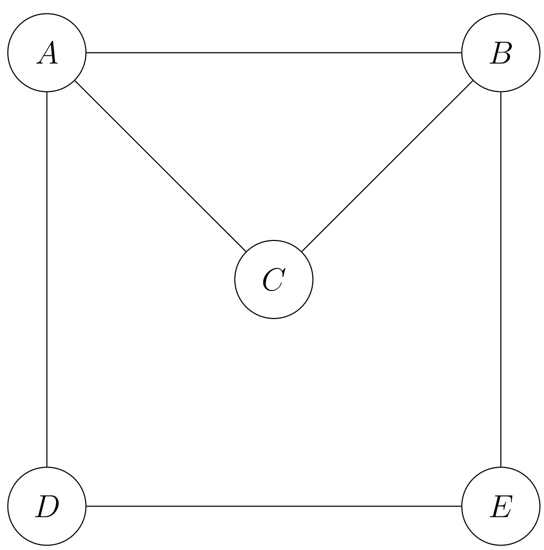
$$p(0,1) = 1, \quad p(i, i+1) = \left(\frac{i+1}{i}\right)^{\alpha} p(i, i-1)$$

for $i \geq 1$.

(a) When $\alpha = 2$, show that $\mathbb{P}_0(X_n^{(2)} \geq 1 \text{ for all } n \geq 1) = 6/\pi^2$.

(b) Calculate $\mathbb{P}_0(X_n^{(\alpha)} \to \infty \text{ as } n \to \infty)$.

3. (Lawler, Exercise 1.8) Consider simple random walk on the graph below. (Recall that simple random walk on a graph is the Markov chain which at each time moves to an adjacent vertex, each adjacent vertex having the same probability).



(a) In the long run, about what fraction of time is spent in vertex A?

(b) Suppose a walker starts in vertex A. What is the expected number of steps until the walker returns to A?

(c) Suppose a walker starts in vertex C. What is the expected number of visits to B before the walker reaches A?

(d) Suppose the walker starts in vertex B. What is the probability that the walker reaches A before the walker reaches C?

(e) Again, assume that walker starts in C. What is the expected number of steps until the walker reaches A?

4. (Lawler, Exercise 1.15) Let X_n be an irreducible Markov chain with state space S starting at state i with transition matrix P. Let

$$T = \min\{n > 0 \, : \, X_n = i\}$$

be the first time that the chain returns to state i. For each state j let $r(j)$ be the expected number of visits to j before returning to i,

$$r(j) = \mathbb{E}\Big[\sum_{n=0}^{T-1} I_{\{X_n=j\}}\Big],$$

where

$$I_{\{X_n=j\}} = \begin{cases} 1, & \text{if } X_n = j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $r(i) = 1$.

(a) Let $\bar{r}$ be the vector whose jth component is $r(j)$. Show that $\bar{r}P = \bar{r}$.

(b) Show that

$$\mathbb{E}[T] = \sum_{j \in S} r(j).$$

(c) Conclude that $\mathbb{E}[T] = \pi(i)^{-1}$, where $\bar{\pi}$ denotes the invariant probability.

5. (Lawler, Exercise 2.2) Consider the following Markov chain with state space $S = \{0, 1, \ldots\}$. A sequence of positive numbers $p_1, p_2, \ldots$ is given with $\sum_{i=1}^{\infty} p_i = 1$. Whenever the chain reaches state 0 it chooses a new state according to p_i. Whenever the chain is in a state other than 0 it proceeds deterministically, one step at a time, toward 0. In other words, the chain has transition probability

$$p(i, i-1) = 1, \quad i > 0,$$
$$p(0, i) = p_i, \quad i > 0.$$

This is a recurrent chain since the chain keeps returning to 0. Under what conditions on the (p_i) is the chain positive recurrent? In this case, what is the limiting probability distribution π? [Hint: it may be easier to compute $\mathbb{E}[T]$ directly where T is the time of the first return to 0 starting from 0.]