Math 285: Winter 2022

Homework 1

Due: Friday, January 14, 11:59 PM

Upload the homework to Gradescope by Friday, January 14, 11:59 PM. Late homework will not be accepted.

1. Consider a rat in a maze consisting of 7 rooms, laid out as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & & \end{bmatrix}$$

Rooms are connected to their vertical and horizontal nearest neighbors only; so 1 is connected to 2 and 4, 2 is connected to 1, 3, and 5, and so forth. At each time $n \in \mathbb{N}$ the rat moves from its current room to one of the adjacent rooms, choosing uniformly among the available choices (the rat changes rooms at each time). Write down the Markov one-step transition matrix for this process.

2. Consider the following simple two stage game consisting of moving between two sites labeled 0 and 1. At each site there is a biased coin with sides labeled 0 and 1. The probability of flipping a 1 at site 0 is $a \in (0,1)$; the probability of flipping a 0 at site 1 is $b \in (0,1)$. If you are at site i at time n, flip the coin at site i; if it comes up i, stay; if it comes up $1-i$, move to site $1-i$. Let X_n be your site at time n. Explain why X_n is a time-homogeneous Markov chain, and calculate the transition probabilities. Then use basic linear algebra techniques to compute $\lim_{n\to\infty} \mathbb{P}(X_n = 0)$. Your answer should be independent of the initial probability distribution of X_0.

3. Let X_0 be a random variable with countable state space S. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, each with uniform distribution on $[0,1]$. Let

$$G \colon S \times [0,1] \to S$$

be a function. Inductively define $X_1, X_2, \ldots$ by

$$X_{n+1} = G(X_n, Y_{n+1}).$$

Show that $(X_n)_{n\geq 0}$ is a Markov chain, and express its transition matrix $\mathbf{P}$ in terms of G. Can all Markov chains be realized in this manner? Can all time-homogeneous Markov chains be realized in this manner?

4. (Lawler, Exercise 1.11) Let $X_1, X_2, \ldots$ be the successive values from independent rolls of a standard six-sided die. Let $S_n = X_1 + \cdots + X_n$. Let

$$T_1 = \min\{n \geq 1 : S_n \text{ is divisible by } 8\},$$

$$T_2 = \min\{n \geq 1 \,:\, S_n - 1 \text{ is divisible by } 8\}.$$

Find $\mathbb{E}(T_1)$ and $\mathbb{E}(T_2)$. (Hint: consider the remainder of S_n after division by 8 as a Markov chain.)

5. (Lawler, Exercise 1.19) Suppose we flip a fair coin repeatedly until we have flipped four consecutive heads. What is the expected number of flips that are needed? (Hint: consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$.)