Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3,4\}$ with states 0 and 4 absorbing, birth rates $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 1$ and death rates $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1$.
   (a) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$.
   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on $\{1,2,3\}$. What is the probability that the process will be absorbed at 0?

Solution.

We have the system of equations

$$u_1 = \frac{1}{3} + \frac{2}{3}u_2 \tag{1}$$
$$u_2 = \frac{1}{3}u_1 + \frac{2}{3}u_3 \tag{2}$$
$$u_3 = \frac{1}{2}u_2 \tag{3}$$

which gives

$$u_2 = \frac{1}{9} + \frac{2}{9}u_2 + \frac{1}{3}u_2 \tag{4}$$
$$\frac{4}{9}u_2 = \frac{1}{9} \tag{5}$$
$$u_2 = \frac{1}{4}, \quad u_1 = \frac{1}{3} + \frac{2}{3}\frac{1}{4} = \frac{6}{12} = \frac{1}{2}, \quad u_3 = \frac{1}{8} \tag{6}$$
$$\frac{1}{3}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{3}\frac{1+2+4}{8} = \frac{7}{24} \tag{7}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. Suppose that the number of printers working at a printing facility at a given time t is given by a continuous time Markov chain $(X_t)_{t\geq 0}$ on the state space $\{0,1,2\}$ with generator

$$Q = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -14 & 4 & 10 \\ 1 & 1 & -11 & 10 \\ 2 & 2 & 8 & -10 \end{array}.$$

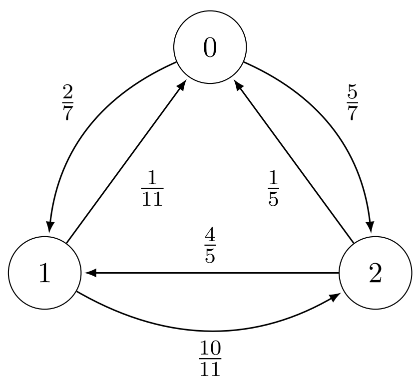
(a) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

(b) Compute the stationary distribution of $(X_t)_{t\geq 0}$. [**Hint.** Remember that there are different ways of finding the stationary distribution].

(c) Suppose that the printing facility works on a 24/7 basis and each printer can produce 100 pages per minute. How many pages does the facility produce on average per minute in the long run?

Solution.

(a) The diagram of the jump chain has the following form



We see that with positive probability the jump chain can transition from any state i to any other state j in one step. Thus, the embedded jump chain is irreducible, and therefore the continuous-time Markov chain $(X_t)_{t\geq 0}$ is irreducible as well.

(b) To compute the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2)$, write the *detailed balance* equation

$$4\pi_0 = \pi_1,$$
$$10\pi_0 = 2\pi_2,$$
$$10\pi_1 = 8\pi_2.$$

The first two equations give

$$\pi_1 = 4\pi_0, \quad \pi_2 = 5\pi_0,$$

which make the third equation redundant. Substituting the above into $\sum_{i=0}^{2} \pi_i = 1$ we get

$$\pi_0 + 4\pi_0 + 5\pi_0 = 10\pi_0 = 1.$$

We conclude that

$$\pi_0 = 0.1, \quad \pi_1 = 0.4, \quad \pi_2 = 0.5.$$

Since $\pi = (0.1, 0.4, 0.5)$ is a solution to the detailed balance equation with strictly positive components, π defines the stationary distribution for the Markov chain $(X_t)_{t\geq 0}$.

(c) In the long run, the entries of the stationary distribution give the average amount of time spent by the Markov chain in each of the states. This means, that in the long run, 10% of the time there are 0 printers working, 40% of the time there is only one printer working, and 50% of the time both printers are working. Therefore, on average in the long run the printing facility produces

$$0.1 \cdot 0 + 0.4 \cdot 100 + 0.5 \cdot 200 = 140$$

pages per minute.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

3. Let $X \sim \text{Unif}[0, Y]$, where Y is a random variable distributed on $[0, 2]$ with linear density

$$f_Y(y) = \alpha y. \tag{8}$$

(a) Determine α.

(b) Compute $E(X \,|\, Y = y)$.

(c) Compute $E(X)$.

Solution.

$$\alpha = \frac{1}{2} \tag{9}$$

$$E(X \,|\, Y = y) = \frac{y}{2} \tag{10}$$

$$E(X) = \int_0^2 \frac{y}{2}\frac{y}{2} dy = \frac{8}{12} = \frac{2}{3} \tag{11}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

4. Suppose that certain company is using age replacement policy for replacing lightbulbs in its offices: a lightbulb is replaced either upon its failure, or after reaching age $T > 0$, whichever comes first. Suppose that each bulb replacement costs 1 dollar, but if it happens due to a failure, then it incurs **additional** costs of 4 dollars per replacement. It is given that the lifetime of a lightbulb has a uniform distribution on the interval $[0, 2]$.

   Determine the optimal replacement age T (that minimizes the long run mean cost of the replacement) and compute the long run mean replacement cost per unit of time for this choice of T. Compare it to the costs of replacement upon failure.

   Solution. Use age replacement strategy from Lecture 20. If the cost of one replacement is K dollars, each replacement due to a failure costs additional c dollars, T is the replacement age and the interrenewal distribution is given by F, then the long run replacement cost (per unit) is given by

$$C(T) = \frac{K + cF(T)}{\int_0^T (1 - F(x))dx}. \tag{12}$$

   In our particular case, $K = 1$, $c = 4$ and

$$F(t) = \begin{cases} 0, & t \le 0, \\ t/2, & 0 < t \le 2, \\ 1, & t > 2, \end{cases} \tag{13}$$

   so

$$\int_0^T (1 - F(x))dx = T - \frac{T^2}{4} \tag{14}$$

   for $0 \le T \le 2$. Therefore,

$$C(T) = \frac{1 + 2T}{T - T^2/4}. \tag{15}$$

   Find the minimum

$$C'(T) = \frac{2(T - T^2/4) - (1 + 2T)(1 - T/2)}{(T - T^2/4)^2} = \frac{T^2/2 + T/2 - 1}{(T - T^2/4)^2} = 0. \tag{16}$$

   Multiplying the numerator by 2, we get that the equation

$$T^2 + T - 2 = 0, \tag{17}$$

   has two solutions, $T = -2$ and $T = 1$. Point $T = 1$ is the point of minimum, therefore, the optimal long run replacement cost is equal to

$$C(1) = \frac{1 + 2}{1 - 1/4} = 4. \tag{18}$$

   The cost of replacement upon failure is $K + c = 1 + 4 = 5$. The age replacement policy with the replacement age $T = 1$ will save the company $5 - 4 = 1$ dollar per each replacement in the long run.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

5. Let ξ_i be independent identically distributed random variables with density

$$f(x) = 2x \tag{19}$$

for $x \in [0, 1]$.

(a) Show that the random process $(X_n)_{n\geq 0}$, given by

$$X_0 = 1, \quad X_n = \frac{3^n}{2^n}\xi_1 \cdots \xi_n, \tag{20}$$

defines a nonnegative martingale.

(b) Estimate the probability that $(X_n)_{n\geq 0}$ ever exceeds 100.

Solution.

(a) Check the definition of a martingale:

$$E(X_n) = \frac{3^n}{2^n}E(\xi_1 \cdots \xi_n) = \frac{3^n}{2^n}\big(E(\xi)\big)^n = 1 < \infty, \tag{21}$$

where we used that

$$E(\xi) = \int_0^1 2x^2 dx = \frac{2}{3}. \tag{22}$$

Moreover,

$$E(X_{n+1}|X_0, \ldots, X_n) = E\Big(\frac{3}{2}\xi_{n+1}X_n|X_0, \ldots, X_n\Big) = \frac{3}{2}E(\xi_{n+1})X_n = X_n. \tag{23}$$

Since $X_n \geq 0$, $(X_n)_{n\geq 0}$ is a nonnegative martingale.

(b) Using the maximal inequality for nonnegative martingales (Lecture 22, page 2)

$$P(\max_{n\geq 0} X_n \geq 100) \leq \frac{E(X_0)}{100} = \frac{1}{100}. \tag{24}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

6. The position of a particle on a real line is described by a Brownian motion starting from 0 with variance parameter $\sigma^2 = 4$ reflected at -10 (boundary/wall at point -10).

   Determine the probability that **at time** $t = 25$ the position of the particle is given by a negative number. [Express the answer in terms of the CDF of the standard normal distribution $\Phi(x)$.]

Solution.

$$\begin{aligned}
P(R_{25} \le 0) &= P(|2B_{25} + 10| - 10 \le 0) && (25)\\
&= P(|10B_1 + 10| \le 10) && (26)\\
&= P(-10 \le 10B_1 + 10 \le 10) && (27)\\
&= P(-2 \le B_1 \le 0) && (28)\\
&= \Phi(0) - (1 - \Phi(2)) \approx 0.498 && (29)
\end{aligned}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

7. Suppose that the price fluctuations of a share are modeled by a Brownian motion starting from $B_0 = 100$ and having variance parameter $\sigma^2 = 4$. What is the probability that the price of the share never drops below 90 dollars on the time interval $[0, 25]$? [Express the answer in terms of the CDF of the standard normal distribution $\Phi(x)$.]

Solution.

$$
\begin{aligned}
P(\min_{0\le t\le 25}(100 + 2B_t) \ge 90) &= P(\min_{0\le t\le 25} B_t \ge -5) && (30)\\
&= P(\max_{0\le t\le 25} B_t \le 5) && (31)\\
&= P(|B_{25}| \le 5) && (32)\\
&= P(|B_1| \le 1) = 2\Phi(1) - 1. && (33)
\end{aligned}
$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)