MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Martingales

Next: PK 8.1

Week 9:

- homework 7 (due Friday, May 27)

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values. For any $\lambda > 0$ and $m \in \mathbb{N}$

$$P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n\geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$. Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

$$1 = \mathbb{1}_{X_0\geq\lambda} + \mathbb{1}_{X_0<\lambda,\, X_1\geq\lambda} + \dots + \mathbb{1}_{X_0<\lambda,\dots,X_{m-1}<\lambda,\, X_m\geq\lambda} + \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda}$$

Compute $E(X_m) = E(X_m \cdot 1)$ using the above partition

Proof of the maximal inequality

$$E(X_m) = \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda}) + E(X_m \underbrace{\mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda}}_{\geq 0})$$

$$\geq \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$ by conditioning on $X_0, X_1, \dots, X_{n-1}, X_n$:

$$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda})$$

$$= E(E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda} \mid X_0,\dots,X_n))$$

$$= E(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda} E(X_m \mid X_0,\dots,X_n))$$

$$= E(X_n \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda, X_n \geq \lambda}) \geq \lambda P(X_0<\lambda,\dots,X_{n-1}<\lambda, X_n\geq\lambda)$$

Sum for all n

$$E(X_m) \geq \lambda \sum_{n=0}^{m} P(X_0<\lambda,\dots,X_{n-1}<\lambda, X_n\geq\lambda) = \lambda P(\max_{0\leq n\leq m} X_n \geq \lambda) \quad \blacksquare$$

Example

A gambler begins with a unit amount of money and faces a series of independent fair games. In each game the gamblers bets fraction p of his current fortune, wins with probability $\frac{1}{2}$, loses with probability $\frac{1}{2}$. Estimate the probability that the gambler ever doubles the initial fortune.

Denote by $Z_n, n \geq 0$, the gambler's fortune after n-th game.

Denote $\{Y_i\}_{i=1}^{\infty}$ i.i.d. r.v.s $P(Y_i = 1+p) = P(Y_i = 1-p) = \frac{1}{2}$

Then $Z_n = Y_1 \cdot Y_2 \cdot \dots \cdot Y_n$, $Z_0 = 1$

$E(Y_i) = (1+p)\frac{1}{2} + (1-p)\frac{1}{2} = 1 \Rightarrow Z_n$ is a nonnegative martingale

$$\Rightarrow P\left(\max_{n \geq 0} Z_n \geq 2\right) \leq \frac{E(Z_0)}{2} = \frac{1}{2}$$

Martingale transform

In the previous example the stake in n-th game is $p Z_{n-1}$. What if we choose another strategy?

<u>Def</u> Let $(X_n)_{n\geq 0}$ be a nonnegative martingale, and let $(C_n)_{n\geq 0}$ be a stochastic process with $C_n = f_n(X_0, \dots, X_{n-1})$. Then the stochastic process

$$\sum_{k=1}^{n} C_k (X_k - X_{k-1}) =: (C \bullet X)_n \quad , \quad (C \bullet X)_0 = 0$$

is called the martingale transform of X by C

Think of
- $X_k - X_{k-1}$ as the winning per unit stake in k-th game
- C_k as your stake in k-th game
  decision is made based on the previous history
- $(C \bullet X)_n$ as total winnings up to time n

Martingale transform

Prop. Let $Z_n = X_0 + (C \bullet X)_n$. Let $C_k > 0$ bounded if $Z_{k-1} > 0$ and $C_k = 0$ if $Z_{k-1} = 0$. Then $(Z_n)_{n \geq 0}$ is a martingale

Proof: $E(Z_{n+1} \mid Z_0, \dots, Z_n) = E(Z_n + C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$

$$= Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid Z_0, \dots, Z_n)$$

Note that $Z_n - Z_{n-1} = C_n (X_n - X_{n-1})$, $Z_0 = X_0$

If $Z_n > 0$, then $C_1 > 0, \dots, C_{n+1} > 0$,

$X_1 = (Z_1 - Z_0) C_1^{-1} + Z_0$, $X_n = (Z_n - Z_{n-1}) C_n^{-1} + X_{n-1}$ and

$$E(Z_{n+1} \mid Z_0, \dots, Z_n) = Z_n + E(C_{n+1}(X_{n+1} - X_n) \mid X_0, \dots, X_n)$$
$$= Z_n + C_{n+1} E(X_{n+1} - X_n \mid X_0, \dots, X_n) = Z_n$$

If $Z_n = 0$, then $C_{n+1} = 0$ and $E(Z_{n+1} \mid Z_0, \dots, Z_n) = 0 = Z_n$

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Gambling example:

Start from the initial fortune $X_0 = 1$. Define

$$Z_n = 1 + (C \cdot X)_n \geq 0$$

fortune after n-th game with strategy C

Then $(Z_n)_{n \geq 0}$ is a nonnegative martingale, $E(Z_0) = 1$

$$\Rightarrow \quad P\left(\max_{n \geq 0} Z_n \geq 2 \right) \leq \frac{1}{2}$$

Convergence of nonnegative martingales

Thm.

If $(X_n)_{n\geq 0}$ is a nonnegative (super)martingale, then with probability 1

$$\exists \lim_{n\to\infty} X_n =: X_\infty$$

and

$$E(X_\infty) \leq E(X_0)$$

Example

An urn initially contains one red ball and one green ball. Choose a ball and return it to the urn together with another ball of the same color. Repeat. Denote by X_n the fraction of red ball after n iterations.

Example (cont.)

(i) $(X_n)_{n\geq 0}$ is a martingale

Denote by R_n the number of red balls after n-th iteration

$$R_n = X_n \cdot (n+2)$$

Then

$$E(X_{n+1} \mid X_0, \dots, X_n) = X_n \frac{R_n + 1}{n+3} + (1 - X_n) \frac{R_n}{n+3}$$

$$= \frac{1}{n+3} (X_n + R_n) = \frac{1}{n+3} \left(X_n + (n+2) X_n \right) = X_n$$

(ii) X_n is nonnegative $\Rightarrow$ $\exists \lim\limits_{n\to\infty} X_n = X_\infty$

(iii) Compute the distribution of X_∞

$$P\left(X_n = \frac{k}{n+2}\right) = \frac{1}{n+1} \quad \text{for } k \in \{1, 2, \dots, n+1\}$$

$$P(X_\infty \leq x) = x \,,\; x \in (0,1) \Rightarrow X_\infty \sim \mathrm{Unif}[0,1)$$