

MATH180C: Introduction to Stochastic Processes II

[Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA](http://math-old.ucsd.edu/~ynemish/teaching/180cA)

[Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB](http://math-old.ucsd.edu/~ynemish/teaching/180cB)

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- homework 5 (due Friday, May 6)
- regrades for Midterm 1 and HW4 active until May 7, 11PM

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i , $X_i \in (0, \infty)$.

Thm.

Proof. $N(t)$ is nondecreasing, therefore

N_∞ is the total number of events ever happened.

Thm (Pointwise renewal thm).

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

Proof. (Only for bounded X_i : $\exists K$ s.t. $P(X_i \leq K) = 1$)

First note that

In lecture 11 we showed that

$$\text{so } M(t) =$$

$$\frac{M(t)}{t} =$$

If $X_i \leq K$, then

Asymptotic distribution of $N(t)$

Thm. Let $N(t)$ be a renewal process with
 $E(X_1) = \mu$ and $\text{Var}(X_1) = \sigma^2$, then

1)

2)

No proof.

Elementary renewal theorem and continuous X_i 's

Two more results (without proofs) about the limiting behaviour of $M(t)$ for models with continuous interrenewal times.

Thm. Let $E(X_1) = \mu$ and let $m(t) = \frac{d}{dt}M(t)$ be the renewal density. Then

Remark $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \alpha$ does not imply in general $\lim_{t \rightarrow \infty} f'(t) = \alpha$

(E.g., take $f(t) = t + \sin t$)

Thm. If additionally $\text{Var}(X_1) = \sigma^2$, then

Example: $X_i \sim \text{Gamma}(2, 1)$

Let $N(t)$ be a renewal process with interrenewal times X_i having Gamma distribution with parameters $(2, 1)$ i.e., $f_{X_i}(t) = t e^{-t}$. Then from the properties of the Gamma distribution (or from direct computations)

$$X_1 + \dots + X_n \sim \text{Gamma}(2n, 1), \text{ so}$$

$$f^{*n}(t) =$$

We can compute the renewal density

$$m(t) =$$

so that $M(t) =$

$$\text{Finally, } E(X_i) = \mu = 2, \text{ Var}(X_i) = \sigma^2 = 2, \text{ so } \frac{\sigma^2 - \mu^2}{2\mu^2} =$$

Joint distribution of age and excess life

From the definition of γ_t and δ_t

$$P(\delta_t \geq x, \gamma_t > y) \quad (x \leq t)$$

=

• Partition wrt the values of $N(t)$

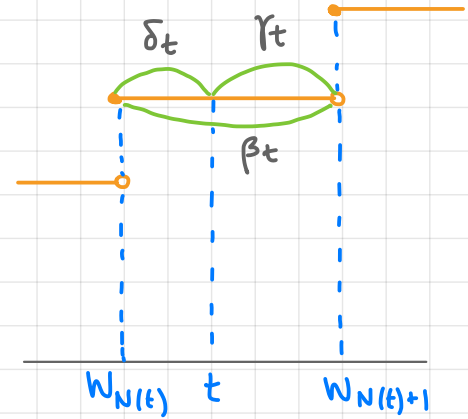
=

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

=

=

=



Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) =$$

=

=

=

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \rightarrow 0$ as $s \rightarrow \infty$ ($\mu = E(X_1)$). Then

$$\lim_{t \rightarrow \infty} P(\delta_t \geq x, \gamma_t > y) =$$

=

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

$$\begin{aligned} (a) \quad P(\gamma_t > y, \delta_t \geq x) &= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u)) dF^{*k}(u) \\ &= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) dM(u) \end{aligned}$$

(b) if additionally the interrenewal times are continuous,

$$\lim_{t \rightarrow \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w)) dw \quad (*)$$

If we denote by $(\gamma_{\infty}, \delta_{\infty})$ a pair of r.v.s with distribution $(*)$

then γ_{∞} and δ_{∞} are continuous r.v.s with densities

$$f_{\gamma_{\infty}}(x) = f_{\delta_{\infty}}(x) =$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on $[0,1]$ (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

↑ renewal equation

E.g.: $M(t) = F(t) + M * F(t),$

$$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_0^{+\infty} g(t-x) dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x) dx$

Def. Function h is called locally bounded if

Def. Function h is absolutely integrable if

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $(*)$, then H is locally bounded and

(b) Conversely, if H is a locally bounded solution to $(*)$, then H is given by $(**)$ [convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there **exists** a **unique** locally bounded solution to $(*)$ given by $(**)$

Examples

- Renewal function: $M(t)$ satisfies

and

$F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

and

(in the Riemann-Stieltjes sense)

f is absolutely integrable, , so

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W'_1, W'_2, \dots)$ with

$$W'_i = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W'
- W'

Example

Example. Compute $\lim_{t \rightarrow \infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then

; if $X_1 < t$ condition on $X_1 = s$

$$E(\gamma_t) =$$

$$E(\gamma_t \mathbb{1}_{X_1 \leq t}) =$$

=

=

=

=

Example (cont)

Assume that $E(X_i) = \mu$, $\text{Var}(X_i) = \sigma^2$

$$E((X_i - t) \mathbb{1}_{X_i > t}) =$$

=

Since we assume that $E(X_i) = \mu$,

and

Finally, we have that

$$H(t) =$$

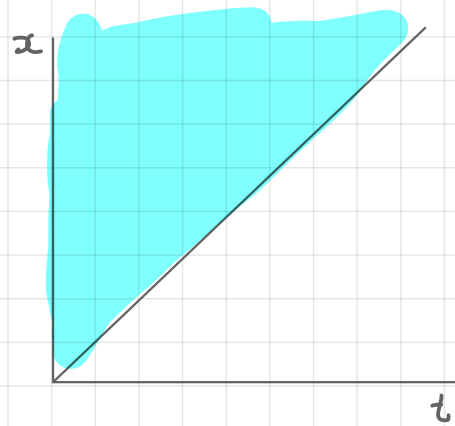
therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)

In particular,

$$\int_0^{\infty} \int_t^{\infty} (1 - F(x)) dx dt =$$



\Rightarrow by part (c) of the key renewal theorem

$$\lim_{t \rightarrow \infty} E(\gamma_t) =$$

Similarly $\lim_{t \rightarrow \infty} E(\delta_t) =$, $\lim_{t \rightarrow \infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_i \sim \text{Unif}[0,1]$

therefore, $\lim_{t \rightarrow \infty} E(Y_t) =$

And the long run expected time between two consecutive earthquakes is