MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Conditioning on continuous random variables
Renewal processes

Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- homework 4 (due Friday, April 29)

Properties of conditional probability / expectation

1) $$P(a < X < b,\ c < Y < d) = \int_c^d \left(\int_a^b f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_c^d P(X \in (a,b) \mid Y = y)\, f_Y(y)\,dy$$

2) $$P(a < X < b) = \int_{-\infty}^{+\infty} \left(\int_a^b f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_{-\infty}^{+\infty} P(X \in (a,b) \mid Y = y)\, f_Y(y)\,dy$$

3) $$E(g(X)) = \int_{-\infty}^{+\infty} E(g(X) \mid Y = y)\, f_Y(y)\,dy$$

Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y) = c_1 E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

5) $E(v(X,Y) \mid Y=y) = E(v(X,y) \mid Y=y)$

In particular, $E(v(X,Y)) = \int_{-\infty}^{+\infty} E(v(X,y) \mid Y=y) f_Y(y)\,dy$

6) $E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) \mid Y=y) f_Y(y)\,dy$

$$= E_Y(h(Y) E(g(X) \mid Y))$$

7) $E(g(X) \mid Y=y) = E(g(X))$ if X and Y are independent

Example 1

Let (X, Y) be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y} - y}, \quad x, y > 0$$

Compute the conditional density of X given $Y=y$.

1) Compute the marginal density of Y

$$f_Y(y) = \int_0^\infty \frac{1}{y} e^{-\frac{x}{y} - y}\, dx = e^{-y} \underbrace{\int_0^\infty \frac{1}{y} e^{-\frac{x}{y}}\, dx}_{\substack{= \\ 1 \text{ for all } y>0}} = e^{-y} \quad (Y \sim Exp(1))$$

2) Compute the conditional density

$$f_{X|Y}(x|y) = \frac{\frac{1}{y} e^{-\frac{x}{y} - y}}{e^{-y}} = \frac{1}{y} e^{-\frac{x}{y}} \quad \Bigg| \Rightarrow \quad \text{Given } Y=y, \; X \sim Exp\left(\frac{1}{y}\right)$$

Example 1 (cont.)

Suppose that $Y \sim Exp(1)$, and X has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

First, $E(X|Y=y)=y$, and using property 3)

$$E(X) = \int_0^\infty E(X|Y=y) f_Y(y)\,dy$$

$$= \int_0^\infty y e^{-y} dy = E(Y) = 1$$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y}-y} dx\,dy = \int_0^\infty \Big(\underbrace{\int_0^\infty \frac{x}{y} e^{-\frac{x}{y}} dx}_{E(X|Y=y)=y} \Big) e^{-y} dy$$

$$= \int_0^\infty y e^{-y} dy = 1$$

Example 2: continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim Unif[0,1]$, $X \sim Bin(N, P)$

What is the distribution of X?

$$P(X=k) = \int_0^1 P(X=k \mid P=s) f_P(s)\, ds$$

$$= \int_0^1 P(X=k \mid P=s)\, ds$$

$$= \int_0^1 \frac{N!}{k!(N-k)!} s^k (1-s)^{N-k}\, ds$$

$$= \frac{N!}{k!(N-k)!} \cdot \frac{k!(N-k)!}{(N+1)!} = \frac{1}{N+1}$$

$\Rightarrow$ X is uniformly distributed $\{0, 1, \dots, N\}$

Example 3

Let X and Y be i.i.d. $\mathrm{Exp}(\lambda)$ r.v.

Define $Z = \frac{X}{Y}$. Compute the density of Z.

- If $X \sim \mathrm{Exp}(\lambda)$, then for $\alpha > 0$ $\alpha X \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$

$$P(\alpha X > t) = P\left(X > \frac{t}{\alpha}\right) = e^{-\lambda \frac{t}{\alpha}} = e^{-\frac{\lambda}{\alpha} \cdot t} \Rightarrow \alpha X \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$$

- $$P(Z > t) = \int_0^\infty P(Z > t \mid Y = y) f_Y(y)\, dy$$

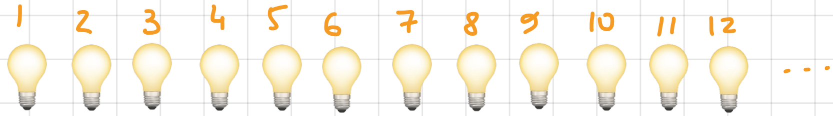
$$= \int_0^\infty P\left(\frac{X}{y} > t\right) \lambda e^{-\lambda y} dy$$

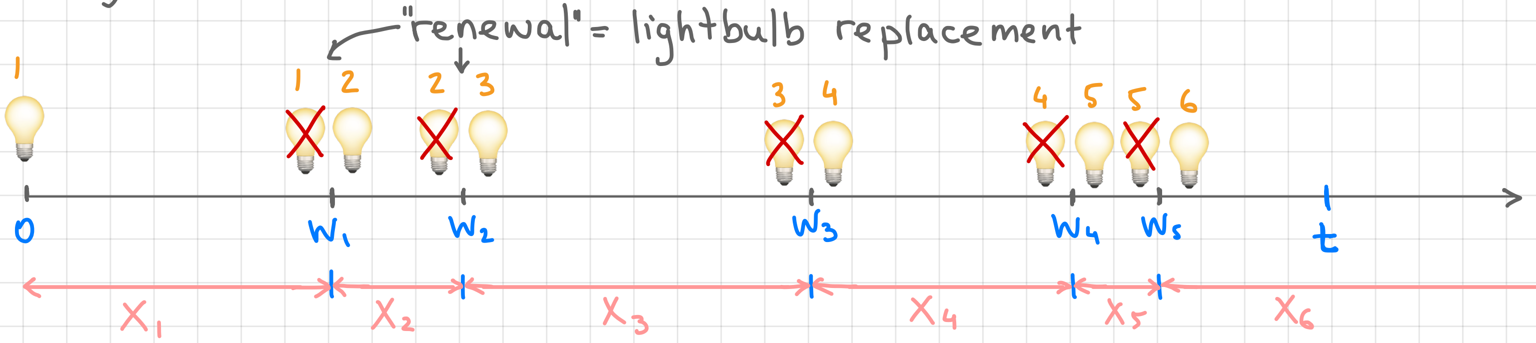
$$= \int_0^\infty e^{-\lambda y t} \lambda e^{-\lambda y} dy = \lambda \int_0^\infty e^{-\lambda y(1+t)} dy = \frac{1}{1+t}$$

$$f_Z(t) = \frac{1}{(1+t)^2}$$

Renewal process

Imagine lightbulbs





X_i – lifetime of the lightbulb #i. W_i = time of i-th "renewal"

Lightbulbs are identical $\Rightarrow$ X_i are i.i.d.

Let $N(t)$ denote the number of renewals up to time t

- What are the properties of $(N(t))_{t\geq 0}$?
- How they depend on the distribution of X_i?

Renewal process. Definition

Def. Let $\{X_i\}_{i\geq 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \cdots + X_n$, $n \geq 1$, and $W_0 := 0$.

We call the counting process

the renewal process.

Remarks. 1) W_n are called the waiting/renewall times

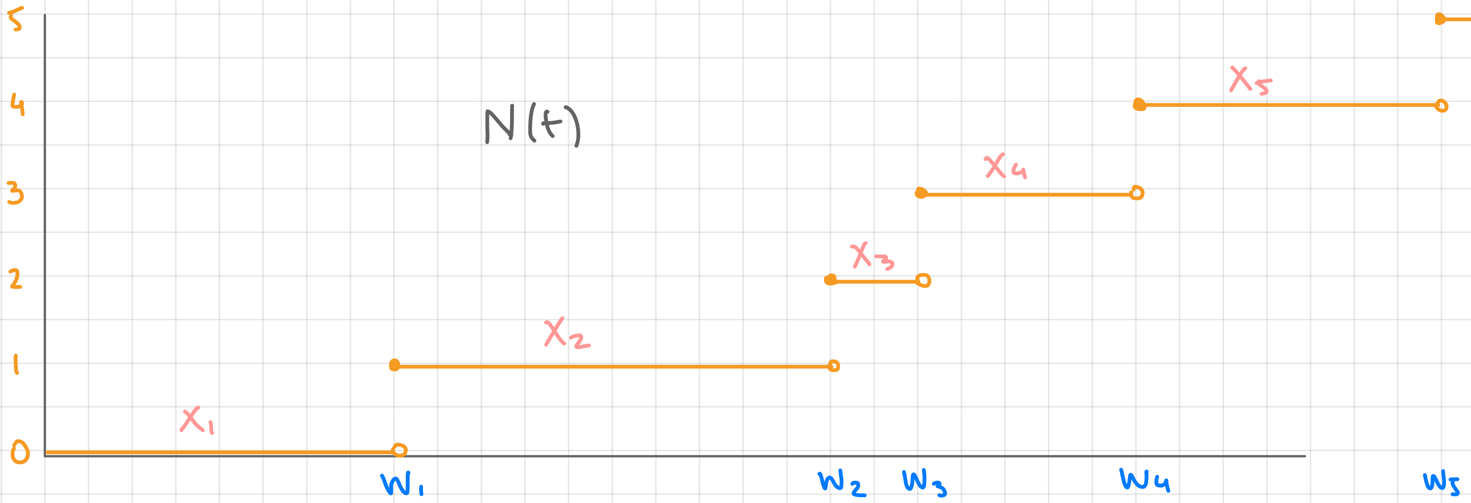
X_i are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \leq a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \leq b\}$$

Renewal process. Definition



Remarks 1) (*) implies that $(N(t))_{t\geq 0}$ is determined by $(W_k)_{k\geq 0}$, so sometimes $(W_k)_{k\geq 0}$ is called renewal process

2) For any t, $W_{N(t)} \leq t < W_{N(t)+1}$

Convolutions of c.d.f.s

Suppose that X and Y are independent r.v.s

$F : \mathbb{R} \to [0,1]$ is the c.d.f. of X (i.e. $P(X \le t) = F(t)$).

$G : \mathbb{R} \to [0,1]$ is the c.d.f. of Y

- if Y is discrete, then

$$F_{X+Y}(t) = P(X+Y \le t) =$$

$$=$$

$$=$$

- if Y is continuous, then

$$F_{X+Y}(t) = P(X+Y \le t) =$$

Distribution of W_k

Let $X_1, X_2, \ldots$ be i.i.d. r.v.s, $X_i > 0$, and let $F : \mathbb{R} \to [0,1]$ be the c.d.f. of X_i (we call F the interoccurrence or interrenewal distribution). Then

- $F_1(t) := F_{W_1}(t) = P(W_1 \le t) = P(X_1 \le t) = F(t)$
- $F_2(t) := F_{W_2}(t) = F_{X_1+X_2}(t) =$
- $F_3(t) := F_{W_3}(t) =$
- More generally,

$$F_n(t) := F_{W_n}(t) = P(W_n \le t) =$$

<u>Remark</u>: $F^{*(n+1)}(t) = \int_0^t F^{*n}(t-x)\, dF(x) = \int_0^t F(t-x)\, dF^{*n}(x)$

Renewal function

Def. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. We call

the renewal function.

Proposition 1. $M(t) =$

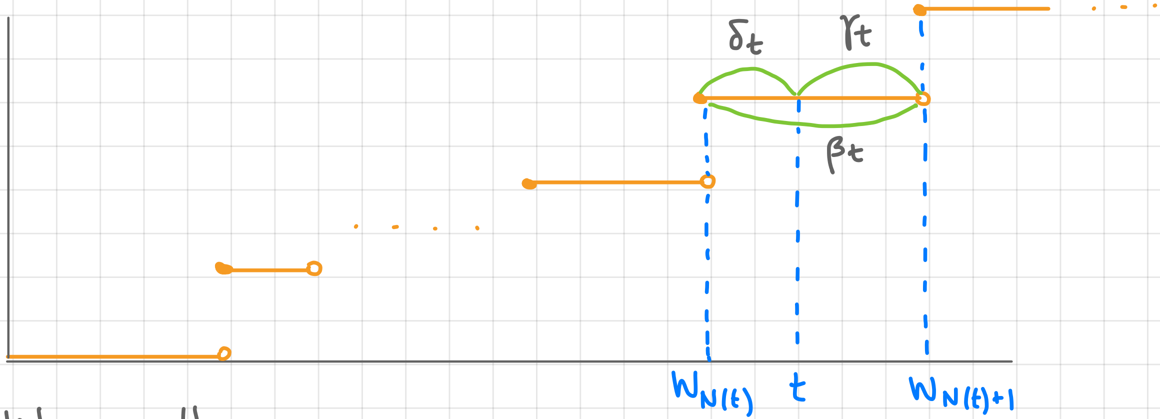
Proof. $M(t) = E(N(t)) =$

$=$

$=$

Related quantities

Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1)

2)

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n \geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\; E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) =$

$E(X_2 + \dots + X_{N(t)+1}) =$

$=$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) =$$

$$=$$

Since $\quad N(t) \geq j-1 \iff W_{j-1} \leq t \iff X_1 + X_2 + \dots + X_{j-1} \leq t$

$$=$$

$$=$$

$$=$$

Remark For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

<u>Proposition 3.</u> Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

<u>Proof.</u> We showed in Proposition 1 that

$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then $M * F =$