☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

1. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & e^{-3t} & \frac{1}{2} - e^{-3t} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 1 & 0 & \frac{1}{2} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 2 & 0 & \frac{1}{2} - \frac{1}{2}e^{-4t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{array}. \tag{1}$$

(a) (7 points) Compute the generator Q of $(X_t)_{t\geq 0}$. [Hint. Recall that $P'(0) = Q$.]

(b) (6 points) Give the jump-and-hold description of $(X_t)_{t\geq 0}$.

(c) (6 points) Draw the rate diagram of $(X_t)_{t\geq 0}$.

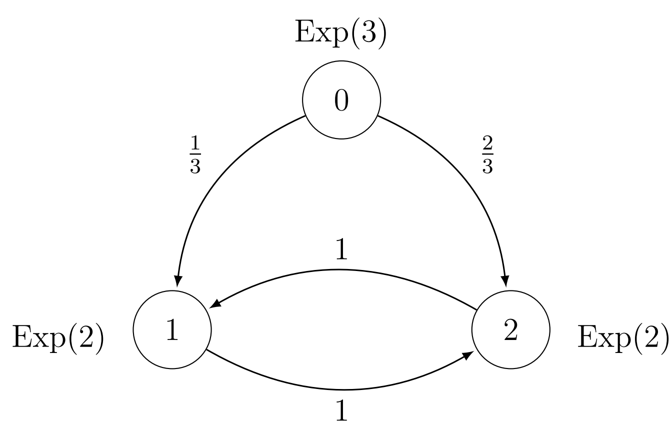
(d) (6 points) Assuming that X_0 is uniformly distributed on $\{0,1,2\}$, compute the probability that $X_1 = 2$.

Solution.

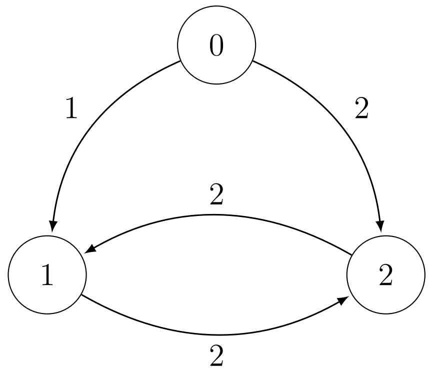
(a) Computing $P'(0)$ gives

$$Q = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -3 & 1 & 2 \\ 1 & 0 & -2 & 2 \\ 2 & 0 & 2 & -2 \end{array} \tag{2}$$

(b) The jump-and-hold diagram



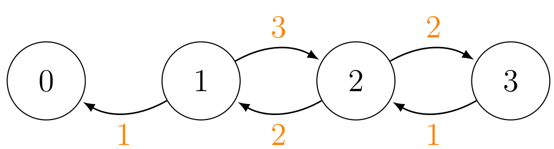
(c) The rate diagram



(d)

$$P(X_1 = 2) = \sum_{i=0}^{2} P(X_1 = 2 \,|\, X_0 = i) P(X_0 = i) \tag{3}$$
$$= \frac{1}{3}\Big(P_{02}(1) + P_{12}(1) + P_{22}(1)\Big) \tag{4}$$
$$= \frac{1}{2} - \frac{1}{6}e^{-4}. \tag{5}$$

2. (25 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on $\{0, 1, 2, 3\}$ described by the following rate diagram



Compute the mean time to absorption starting from state 1 (i.e., given $X_0 = 1$).

Solution.

This chain has three transient states $\{1, 2, 3\}$ and one absorbing state 0. Denote by T the absorption time $T = \min\{t : X_t = 0\}$ and denote by w_i the expected time to absorption starting from state i, i.e.,

$$w_i := E(T \,|\, X_0 = i). \tag{6}$$

In order to solve this problem we have to compute w_1.

Unknown quantities (w_1, w_2, w_3) satisfy the following system of equations:

$$w_1 = \frac{1}{4} + \frac{3}{4}w_2, \tag{7}$$
$$w_2 = \frac{1}{4} + \frac{1}{2}w_1 + \frac{1}{2}w_3, \tag{8}$$
$$w_3 = 1 + w_2. \tag{9}$$

Plugging the first and the third equations into the second equation gives

$$w_2 = \frac{1}{4} + \frac{1}{8} + \frac{3}{8}w_2 + \frac{1}{2} + \frac{1}{2}w_2 = \frac{7}{8} + \frac{7}{8}w_2, \tag{10}$$

from which $w_2 = 7$. Plugging this into the first equation gives the final answer $w_1 = \frac{11}{2}$.

3. (25 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0, 1, 2\}$ with generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\Vert \begin{array}{ccc} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 0 & 1 & -1 \end{array} \right\Vert \end{array}. \tag{11}$$

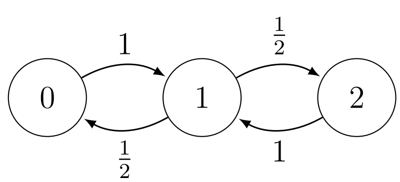
(a) (10 points) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

(b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

(c) (5 points) What is the expected average fraction of time that $(X_t)_{t\geq 0}$ will spend in states 1 and 2 in the long run?

Solution.

(a) The diagram of the jump chain is



All states communicate, therefore the jump chain is irreducible. This implies that $(X_t)_{t\geq 0}$ is also irreducible.

(b) Denote by (π_0, π_1, π_2) the limiting distribution. Then (π_0, π_1, π_2) satisfy the following system

$$-2\pi_0 + 3\pi_1 = 0, \tag{12}$$
$$2\pi_0 - 6\pi_1 + \pi_2 = 0, \tag{13}$$
$$3\pi_1 - \pi_2 = 0, \tag{14}$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{15}$$

The first equation gives $\pi_0 = \frac{3}{2}\pi_1$, the third equation gives $\pi_2 = 3\pi_1$. Plugging this into the last equation gives

$$\pi_1\left(\frac{3}{2} + 1 + 3\right) = \pi_1\frac{11}{2} = 1, \tag{16}$$

so

$$\pi_1 = \frac{2}{11}, \quad \pi_0 = \frac{3}{11}, \quad \pi_2 = \frac{6}{11}. \tag{17}$$

(c) In the long run, the process will spend $\pi_1 + \pi_2 = \frac{8}{11}$ of time in states 1 and 2.

4. (25 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

Let X_t denote the number of printers in operating state at time t.

(a) (10 points) Assuming without proof that $(X_t)_{t\geq 0}$ is a Markov process, determine the generator of $(X_t)_{t\geq 0}$ (you can provide rigorous computations for only one entry of matrix Q.)

[Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \to 0$.]

(b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

(c) (5 points) In the long run, how many pages does the facility produce on average per minute?

Solution.

(a) The generator of $(X_t)_{t\geq 0}$ is given by

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} -1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 1 & -1 \end{array} \right\| \end{array}. \tag{18}$$

Example of computations of q_{ij}.

- $X_0 = 1$ means that one printer is operating and one printer is broken.
  - $X_h = 0$ means that the operating printer stopped working before time h and the broken printed was not repaired before time h, so

$$P(X_h = 0 \,|\, X_0 = 1) \tag{19}$$
$$= P(\text{printer's working time} \leq h) P(\text{printer's repair time} > h) \tag{20}$$
$$= (1 - e^{-\frac{1}{2}h}) e^{-h} = \frac{1}{2}h + o(h) \tag{21}$$

and $q_{10} = \frac{1}{2}$.

- $X_h = 2$ means that the broken printer got repared before time h and the operating printer was is still working at time h, so

$$\begin{aligned} &P(X_h = 2 \,|\, X_0 = 1) && (22)\\ &\quad = P(\text{repare time} \leq h) P(\text{working time} > h) && (23)\\ &\quad = (1 - e^{-h}) e^{-\frac{1}{2}h} = h + o(h) && (24)\end{aligned}$$

and $q_{12} = 1$.

– $X_0 = 2$ means that both printers are working

- $X_h = 1$ means that one of the two printers stopped working before time h and the other is working at time h (note that there are two choices of which of the two got broken), so

$$P(X_h = 1 \,|\, X_0 = 2) \tag{25}$$

$$= 2P(\text{printer's working time} \leq h) P(\text{printer's working time} > h) \tag{26}$$

$$= 2(1 - e^{-\frac{1}{2}h}) e^{-\frac{1}{2}h} = 2\Big(\frac{1}{2}h + o(h)\Big) = h + o(h) \tag{27}$$

and $q_{21} = 1$.

(b) Let (π_0, π_1, π_2) be the stationary distribution. Then (π_0, π_1, π_2) should satisfy the following system

$$-\pi_0 + \frac{1}{2}\pi_1 = 0, \tag{28}$$

$$\pi_0 - \frac{3}{2}\pi_1 + \pi_2 = 0, \tag{29}$$

$$\pi_1 - \pi_2 = 0, \tag{30}$$

$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{31}$$

From the first and the third equations we have that $\pi_0 = \frac{1}{2}\pi_1$ and $\pi_2 = \pi_1$. Plugging this into the last equation gives

$$\pi_1\Big(\frac{1}{2} + 1 + 1\Big) = \frac{5}{2}\pi_1 = 1, \tag{32}$$

from which we get that

$$\pi_0 = \frac{1}{5}, \quad \pi_1 = \frac{2}{5}, \quad \pi_2 = \frac{2}{5}. \tag{33}$$

(c) In the long run, $\frac{1}{5}$ of the time both printers are broken printing 0 pages per minute, $\frac{2}{5}$ of the time only one printer is working producing 100 pages per minute and $\frac{2}{5}$ of the time both printer are working producing 200 pages per minute. Therefore, on average the printing facility produces

$$\frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 200 = 120 \tag{34}$$

pages per minute.