Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

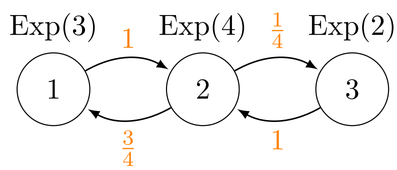
☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. Certain solar power plant has three operating modes: low intensity, medium intensity and high intensity. The transitions from one operating mode to another form a continuous time Markov chain on the states $\{L, M, H\}$ with generator $Q = \left(\begin{array}{c|ccc} & L & M & H \\ \hline L & -3 & 3 & 0 \\ M & 3 & -4 & 1 \\ H & 0 & 2 & -2 \end{array}\right)$. Denote this Markov chain by $(X_t)_{t\geq 0}$
   (a) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ 0 and explain why $(X_t)_{t\geq 0}$ is irreducible.
   (b) Compute the stationary distribution for $(X_t)_{t\geq 0}$.
   (c) What is the expected average fraction of time that the plant spends in the low intensity mode in the long run?

Solution.

(a) The parameters of the jump-and-hold diagram can be read off from the generator matrix Q



(b) The stationary distribution $\pi = (\pi_L, \pi_M, \pi_H)$ is determined from the equations $\pi Q = 0$ and $\pi_L + \pi_M + \pi_H = 1$.

$$-3\pi_L + 3\pi_M = 0, \quad (1)$$
$$3\pi_L - 4\pi_M + 2\pi_H = 0, \quad (2)$$
$$\pi_M - 2\pi_H = 0, \quad (3)$$
$$\pi_L + \pi_M + \pi_H = 1. \quad (4)$$

The first and third equations give $\pi_L = \pi_M$ and $\pi_M = 2\pi_H$. Plugging this into the last equation gives

$$\pi_H = 0.2, \quad \pi_M = 0.4, \quad \pi_L = 0.4. \quad (5)$$

(c) The average fraction of time spent in the low intensity state in the long run is given by (see lecture 10, page 11) $\pi_L = 0.4$.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c|c|c|c} & 0 & 1 & 2 \\ 0 & \frac{1}{3} + e^{-4t} - \frac{1}{3}e^{-3t} & \frac{1}{6} - \frac{1}{2}e^{-4t} + \frac{1}{3}e^{-3t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 1 & \frac{1}{3} + e^{-4t} - \frac{4}{3}e^{-3t} & \frac{1}{6} - \frac{1}{2}e^{-4t} + \frac{4}{3}e^{-3t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 2 & \frac{1}{3} - e^{-4t} + \frac{2}{3}e^{-3t} & \frac{1}{6} + \frac{1}{2}e^{-4t} - \frac{2}{3}e^{-3t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{array}.$$

(a) Determine the distribution of the sojourn times of the process at states 0, 1 and 2.

(b) In the long run, what fraction of time will the process $(X_t)_{t\geq 0}$ spend in state 0? [**Hint.** You can answer this question without solving any equations, and if you do so you should clearly state which results you use.]

(c) Let $Q = (q_{ij})_{i,j=0}^{2}$ be the generator matrix of $(X_t)_{t\geq 0}$. Compute q_{10}. Suppose you observe the process jumping from state 2 to state 0. What is the average time that you have to wait until the next time you observe the jump from state 2 to state 0?

Solution.

(a) The distribution of the sojourn times can be read off from the infinitesimal generator Q, and from the relation between the Markov semigroup $P(t)$ and Q we have that $Q = P'(0)$. Therefore, to determine the distribution of the sojourn times it is enough to compute the derivatives of the diagonal entries of $P(t)$ at $t = 0$

$$P'_{00}(0) = -3, \quad P'_{11}(0) = -2, \quad P'_{22}(0) = -2.$$

Thus, the sojourn times at states 0, 1 and 2 have exponential distributions with rates $q_0 = 3$, $q_1 = 2$, $q_2 = 2$ correspondingly.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution for the Markov chain $(X_t)_{t\geq 0}$. Then π_i, $i \in \{0,1,2\}$, gives the average long run fraction of time spent by the process in state i.

In order to compute π_0, note that from the theorem about the long run behavior of continuous time Markov chains, $P_{i0} \to \pi_0$ as $t \to \infty$. If we take the limit in the above explicit formula for $P(t)$ we get

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}, \tag{6}$$

and thus on average in the long run the process spends 1/3 of the time in state 0.

(c) If Q is the infinitesimal generator of $(X_t)_{t\geq 0}$, then

$$q_{10} = P'_{10}(0) = 0.$$

In particular this means that the process cannot jump directly from state 1 to state 0; the process can jump to state 0 only from state 2.

In order to compute the average time required to observe the transition from 2 to 0 happening again, we can either apply the first step analysis, or use the theorem about the long run behavior of the continuous time Markov chains. I present below the second solution.

From the theorem about the long run behavior of the continuous time Markov chains,

$$\pi_i = \frac{1}{q_i m_i},$$

where m_i is the average return time to state i. From this we have that the average return time to 0 is given by

$$m_0 = \frac{1}{q_0 \pi_0} = \frac{1}{\frac{1}{3} 3} = 1.$$

If you observe the transition from state 2 to state 0, then the return of the process to state 0 can only occur through a transition from 2 to 0 ($q_{10} = 0$, so the jumps from 1 to 0 are not allowed). Therefore, the average time to see again the transition from 2 to 0 is equal to $m_0 = 1$.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

3. Let X and Y be two random variables. Suppose that Y has exponential distribution with rate $\lambda > 0$, and suppose that given $Y = y$, $y > 0$, the random variable X has normal distribution with mean y and variance 1.
   (a) Compute $E(X)$.
   (b) Compute $P(X > Y)$.

Solution.

(a) Compute $E(X)$ by conditioning on the value of Y:

$$E(X) = \int_0^\infty E(X \mid Y = y)\lambda e^{-\lambda y} dy = \int_0^\infty y\lambda e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{7}$$

(b) Compute $P(X > Y)$ by conditioning on the value of Y:

$$P(X > Y) = \int_0^\infty P(X > Y \mid Y = y)\lambda e^{-\lambda y} dy = \int_0^\infty \frac{1}{2}\lambda e^{-\lambda y} dy = \frac{1}{2}. \tag{8}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

4. The time intervals between two consecutive rainstorms in San Diego are independent identically distributed random variables with density (in years)

$$f(x) = \begin{cases} 2x, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

(a) Compute the long run expected time between the last rainstorm and the next rainstorm.

(b) What is the long run probability that it has been at most 6 months since the last rainstorm?

Solution.

(a) If $\delta(t)$ is the current life (age) of the renewal process at time t (time from the last rainstorm to time t), and $\gamma(t)$ is the residual life of the renewal process at time t (time until the next rainstorm after time t), then we have to compute

$$\lim_{t\to\infty} E(\delta(t) + \gamma(t)) = \lim_{t\to\infty} E(\beta(t)). \tag{10}$$

Lecture 19, page 3:

$$\lim_{t\to\infty} E(\beta(t)) = \frac{\sigma^2 + \mu^2}{\mu}, \tag{11}$$

where μ and σ^2 are the mean and variance of the interrenewal times.

$$\mu = \int_0^1 2x^2 dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}, \tag{12}$$

$$\mu^2 + \sigma^2 = \int_0^1 2x^3 dx = \frac{x^4}{2}\Big|_0^1 = \frac{1}{2}, \tag{13}$$

therefore

$$\lim_{t\to\infty} E(\beta(t)) = \frac{3}{4}. \tag{14}$$

(b) In terms of the renewal process, the long run probability that it has been at most 6 months since the last rainstorm is given by

$$\lim_{t\to\infty} P(\delta(t) < 1/2). \tag{15}$$

Lecture 17, page 4:

$$\lim_{t\to\infty} P(\delta(t) < 1/2) = \int_0^{1/2} \frac{1}{\mu}(1 - F(x))dx, \tag{16}$$

where $F(x)$ is the interrenewal distribution. Note, that $F(x) = 1$ for $x \geq 1$. For $x \in (0,1)$

$$F(x) = \int_0^x 2s ds = x^2. \tag{17}$$

Therefore,

$$\lim_{t\to\infty} P(\delta(t) < 1/2) = \int_0^{1/2} \frac{3}{2}(1 - x^2)dx = \frac{3}{2}\Big(x - \frac{x^3}{3}\Big)\Big|_0^{1/2} = \frac{11}{16}. \tag{18}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

5. Let ξ_i be independent indentically distributed random variables having normal distribution $N(0,4)$ with mean zero and variance 4.

   (a) Show that the random variable $(X_n)_{n\geq 0}$, given by
   $$X_0 = 1, \quad X_n = \frac{1}{4^n}\xi_1^2\cdots\xi_n^2,$$
   defines a nonnegative martingale.

   (b) Estimate the probability that $(X_n)_{n\geq 0}$ ever exceeds 100.

Solution.

(a) Check the definition of a martingale:
$$E(|X_n|) = \frac{1}{4^n}E(\xi_1^2\cdots\xi_n^2) = \frac{1}{4^n}(E(\xi^2))^n = 1 < \infty, \tag{19}$$
$$E(X_{n+1}|X_0,\ldots,X_n) = E\Big(\frac{\xi_{n+1}^2}{4}X_n|X_0,\ldots,X_n\Big) = E\Big(\frac{\xi_{n+1}^2}{4}\Big)X_n = X_n. \tag{20}$$
Since $X_n \geq 0$, $(X_n)_{n\geq 0}$ is a nonnegative martingale.

(b) Using the maximal inequality for nonnegative martingales (Lecture 22, page 2)
$$P(\max_{n\geq 0} X_n \geq 100) \leq \frac{E(X_0)}{100} = \frac{1}{100}. \tag{21}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

6. Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance parameter σ^2. It is given that $X_0 = 0$, $E(X_1) = \frac{1}{2}$ and $\text{Var}(X_1) = 1$.

   (a) Determine μ and σ^2.

   (b) Suppose that the price fluctuations of a share are modeled by the process $(Z_t)_{t\geq 0}$ given by

$$Z_t = e^{X_t}. \tag{22}$$

   Determine the probability that the price of the share triples before it drops by two thirds (i.e., probability that the price increases from 1 to 3 before in drops from 1 to 1/3).

Solution.

(a) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then $E(X_t) = \mu t$ and $\text{Var}(X_t) = \sigma^2$, therefore we conclude that $\mu = 1/2$ and $\sigma^2 = 1$.

(b) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then the process $(Z_t)_{t\geq 0}$ given by $Z_t = e^{X_t}$ is a geometric Brownian motion with drift α, where

$$\alpha = \mu + \sigma^2/2 = 1. \tag{23}$$

Denote $T := \min\{t : Z_t = 3 \text{ or } Z_t = 1/3\}$. Compute

$$1 - \frac{2\alpha}{\sigma^2} = 1 - \frac{2}{1} = -1. \tag{24}$$

Using the "gambler's ruin" theorem for geometric Brownian motion (Lecture 27, page 13)

$$P(Z_T = 3) = \frac{1 - (1/3)^{-1}}{3^{-1} - (1/3)^{-1}} = \frac{1 - 3}{1/3 - 3} = \frac{3}{4}. \tag{25}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

7. Let τ_1 be the smallest zero of a standard Brownian motion that exceeds $b > 0$. Compute $P(\tau_1 < t)$ for $t > b$.

   Solution. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $t > b$. The event $\{\tau_1 < t\}$ means that B has a zero on the interval (b, t). Using the theorem about the distribution of zeros of Brownian motion (lecture 26, page 4) we get that

$$P(\tau_1 < t) = P(B \text{ has zero on } (b,t)) = \frac{2}{\pi} \arccos \sqrt{b/t}. \tag{26}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

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