MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: FSA for general MC

Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- homework 2 (due Friday April 15)
- HW 1 regrades: Wednesday April 13

Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Longrightarrow$ there is a Q-matrix Q such that

$$P(t) = e^{tQ}$$

$p_{ij}(h) = q_{ij} h + o(h) \quad i \neq j$

$p_{ii}(h) = 1 + q_{ii} h + o(h)$

In particular,

$$P(h) = I + Qh + o(h) \quad \text{as } h \to 0$$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

Q is called the infinitesimal generator of $(X_t)_{t \geq 0}$

Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^{N}$ be a Q-matrix. Denote $q_i = \sum_{j \neq i} q_{ij}$

so that

$$Q = \begin{pmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & & \end{pmatrix} \qquad q_0 = \sum_{i \neq 0} q_{0i} \quad \vdots$$

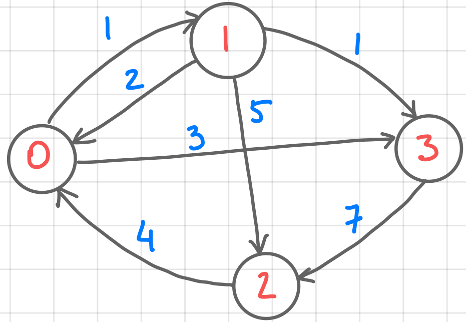
Denote $Y_k := X_{W_k}$ (jump chain).

Then the MC with generator matrix Q has the following equivalent jump and hold description

- sojourn times S_k are independent r.v. with $P(S_k > t \mid Y_k = i) = e^{-q_i t}$ $\left(S_k \sim \mathrm{Exp}(q_i) \right)$
- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) = \dfrac{q_{ij}}{q_i}$

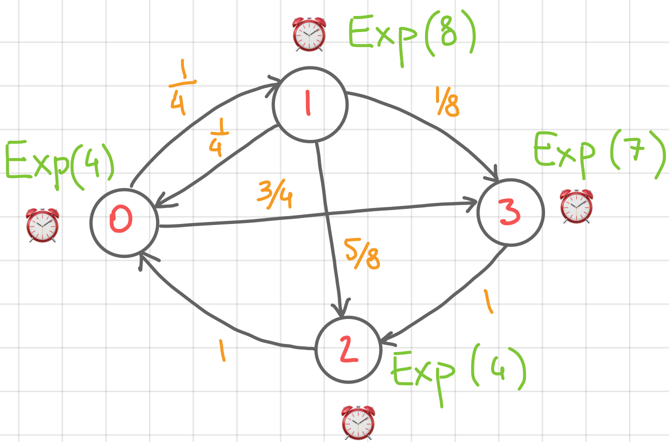
Example

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & & & & \\
1 & & & & \\
2 & & & & \\
3 & & & &
\end{array}
\quad
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{pmatrix}
-4 & 1 & 0 & 3 \\
2 & -8 & 5 & 1 \\
4 & 0 & -4 & 0 \\
0 & 0 & 7 & -7
\end{pmatrix}
$$



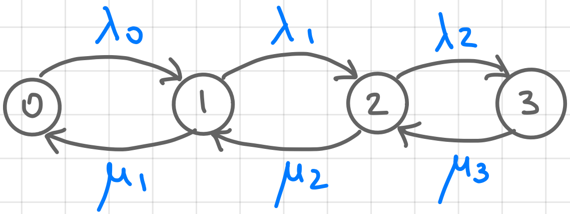
$i \xrightarrow{\alpha} j$ "=" $P_{ij}(h) = \alpha h + o(h)$

as $h \to 0$

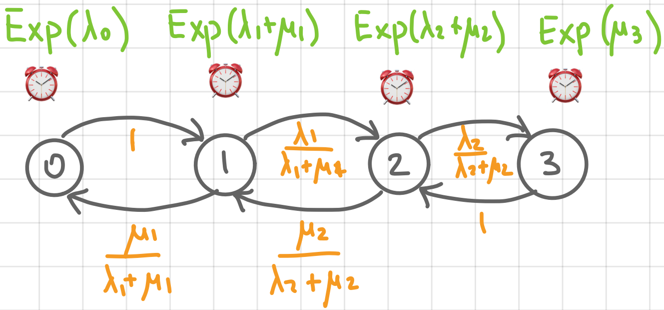


Example

Birth and death process on $\{0,1,2,3\}$

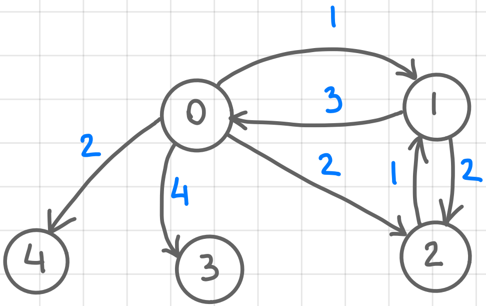


$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \mu_3 & -\mu_3 \end{pmatrix}$$



General continuous time finite state MCs

Rate diagram



Generator

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -9 & 1 & 2 & 4 & 2 \\ 1 & 3 & -5 & 2 & & \\ 2 & & 1 & -1 & & \\ 3 & & & & 0 & \\ 4 & & & & & 0 \end{array}$$

Infinitesimal description

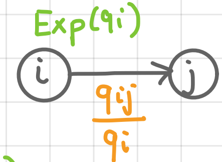
$$P_{ij}(h) = q_{ij}h + o(h) \quad , i \neq j$$

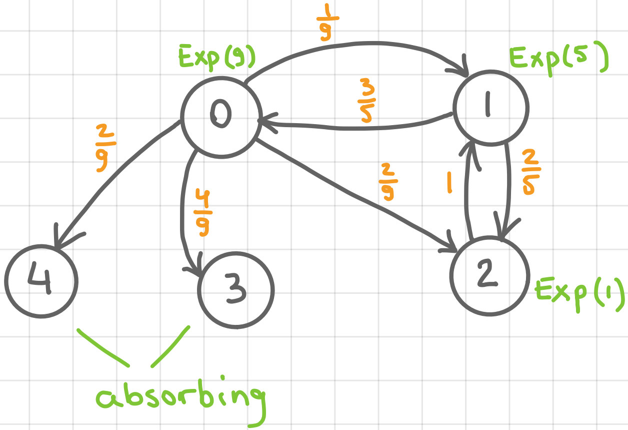
$$P_{ii}(h) = 1 - q_i h + o(h)$$

$$P_{02}(h) = 2h + o(h)$$

$$P_{00}(h) = 1 - 9h + o(h)$$

Jump and hold





Absorption probabilities for finite state chains

By considering the jump chain $(Y_n)_{n\geq 0}$ with $Y_n = X_{W_n}$ and its transition probabilities $P(Y_{n+1}=j \mid Y_n = i) = \frac{q_{ij}}{q_i}$ we can apply the first step analysis to compute, e.g., the absorption probabilities (similarly as for B&D)

If state i is absorbing, then $q_{ij} = 0$ for all $j \neq i$ (no jumps from state i), so $q_i = q_{ii} = 0$. Let Q be given by

$$Q = \begin{array}{c|cccccc} & 0 & \cdots & K-1 & K & \cdots & N \\ \hline 0 & -q_0 & & & \cdots & q_{ij} & \\ \vdots & \vdots & & & & \vdots & \\ K-1 & q_{ij} & \cdots & -q_{K-1} & & & \\ K & & & & 0 & & 0 \\ \vdots & & 0 & & & \ddots & \\ N & & & & & & 0 \end{array}$$

with $\{0,\dots,K-1\}$ transient,
$\{K,\dots,N\}$ absorbing

Absorption probabilities for finite state chains

$$Q = \begin{array}{c|cccccc} & 0 & \cdots & K-1 & K & \cdots & N \\ \hline 0 & -q_0 & & & \cdots & q_{ij} & \\ \vdots & \vdots & & & & \vdots & \\ K-1 & q_{ij} & \cdots & -q_{K-1} & & & \\ K & & & & 0 & & 0 \\ \vdots & & 0 & & & \ddots & \\ N & & & & & & 0 \end{array}$$

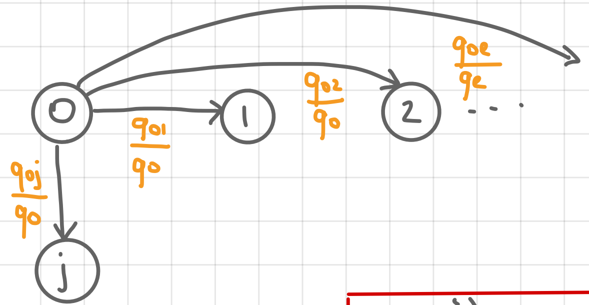
Let $i \in \{0, \dots, K-1\}$, $j \in \{K, \dots, N\}$.

Let $\mathbf{M} = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid X_0 = i)$.

Then FSA leads to the system

Jump chain



$$u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid Y_0 = i)$$

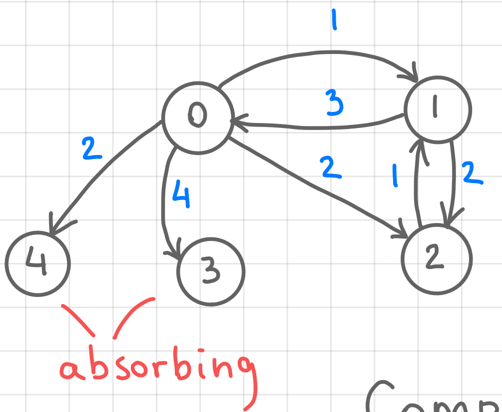
$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{N} P(Y_M = j \mid Y_0 = i, Y_1 = \ell)\, P(Y_1 = \ell \mid Y_0 = i)$$

$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \underbrace{P(Y_M = j \mid Y_1 = \ell)}_{\text{SMP} \quad = u_\ell^{(j)}} \frac{q_{i\ell}}{q_i} + \underbrace{P(Y_M = j \mid Y_1 = j)}_{=1} \frac{q_{ij}}{q_i}$$

$$\boxed{u_i^{(j)} = \underbrace{\frac{q_{ij}}{q_i}}_{P(Y_{n+1}=j \mid Y_n = i)} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \underbrace{\frac{q_{i\ell}}{q_i}}_{P(Y_{n+1}=\ell \mid Y_n = i)} u_\ell^{(j)}}$$

Example

Rate diagram



Generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Compute $P(Y_M = 3)$ if $P(X_0 = i) = p_i$ for $i = 0,1,2$

$\sum p_i = 1$

Denote $u_i = P(Y_M = 3 \mid Y_0 = i)$.

$$\begin{cases} u_0 = \frac{1}{9}u_1 + \frac{2}{9}u_2 + \frac{4}{9} = \frac{903}{90} \\ u_1 = \frac{3}{5}u_0 + \frac{2}{5}u_2 \\ u_2 = u_1 \end{cases} \qquad \begin{cases} 9u_0 = u_0 + 2u_0 + 4, \quad u_0 = u_1 = u_2 = \frac{2}{3} \\ u_1 = u_0 \\ u_2 = u_1, \end{cases}$$

$$P(Y_M = 3) = \sum_{i=0}^{2} \overbrace{P(Y_M = 3 \mid Y_0 = i)}^{u_i} P(Y_0 = i) = \sum_{i=0}^{2} \frac{2}{3} p_i = \frac{2}{3}$$

Mean time to absorption

Similar analysis as was applied to B&D processes can be used to compute the mean time to absorption: before each jump from step i to state j the process sojourns $\frac{1}{q_i}$ on average in state i.

$$
Q = \begin{array}{c} \\ 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array}
\begin{array}{c}
\begin{array}{ccc|ccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\
\left(\begin{array}{ccc|ccc}
-q_0 & & & & \cdots & q_{ij} \\
\vdots & & & & & \vdots \\
q_{ij} & \cdots & -q_{K-1} & & & \\
\hline
 & & & 0 & & 0 \\
 & 0 & & & \ddots & \\
 & & & & & 0
\end{array}\right)
\end{array}
$$

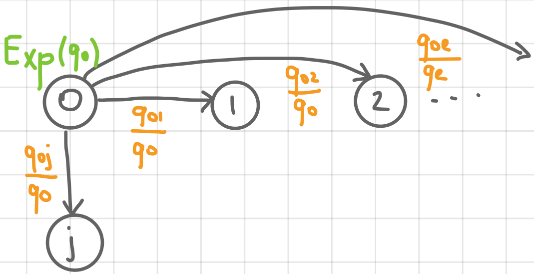
Let $T = \min\{t : X_t \in \{K, \dots, N\}\}$

$M = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $w_i = E(T \mid X_0 = i)$

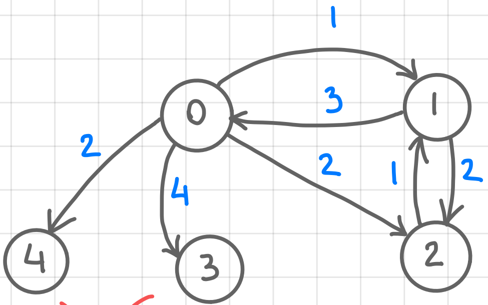
Then FSA gives

$$
\boxed{w_i = \frac{1}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \frac{q_{i\ell}}{q_i} w_\ell}
$$



Example

Rate diagram



absorbing

Generator

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -9 & 1 & 2 & 4 & 2 \\ 1 & 3 & -5 & 2 & & \\ 2 & & 1 & -1 & & \\ 3 & & & & 0 & \\ 4 & & & & & 0 \end{array}$$

$$T = \min\{t : X_t \in \{3,4\}\}$$

$$w_i = E(T \mid X_0 = i)$$

$$\begin{cases} w_0 = \frac{1}{9} + \frac{1}{9} w_1 + \frac{2}{9} w_2 \\ w_1 = \frac{1}{5} + \frac{3}{5} w_0 + \frac{2}{5} w_2 \\ w_2 = 1 + 1 \cdot w_1 \end{cases}$$

$$\begin{cases} w_0 = 1 \\ w_1 = 2 \\ w_2 = 1 + w_1 \qquad w_2 = 3 \end{cases}$$

Kolmogorov equations

Jump and hold description is very intuitive, gives a very clear picture of the process, but does not answer to some very basic questions, e.g., computing $P_{ij}(t) := P(X_t = j \mid X_0 = i)$.

For computing the transition probabilities the differential equation approach is more appropriate.

In order to derive the system of differential equations for $P_{ij}(t)$ from the infinitesimal description, we start from the familiar relation:

Chapman-Kolmogorov equation (semigroup property)

Chapman-Kolmogorov equation

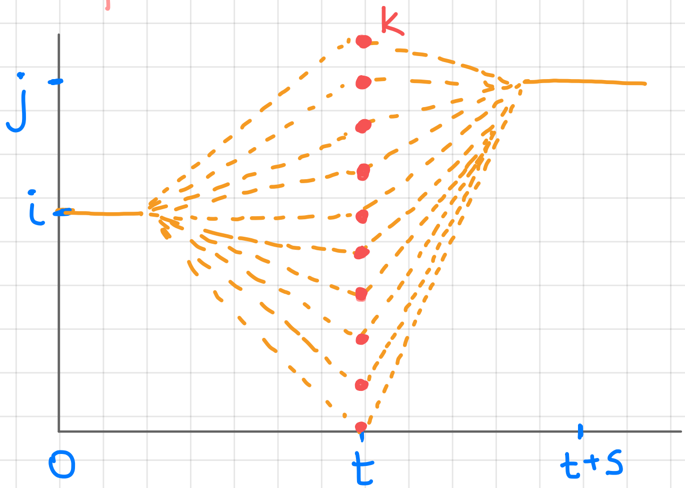
$$P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)$$

condition on the value of X_t

$$= \sum_{k=0}^{N} P(X_{t+s} = j \mid X_0 = i, X_t = k)\, P(X_t = k \mid X_0 = i)$$

Markov $= \sum_{k=0}^{N} P(X_{t+s} = j \mid X_t = k)\, P(X_t = k \mid X_0 = i)$

stationary trans. prob. $= \sum_{k=0}^{N} P(X_s = j \mid X_0 = k)\, P(X_t = k \mid X_0 = i) = \sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$



Or in matrix form

$$P(t+s) = P(t)P(s)$$