MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 10:

- homework 8 (due Friday, June 3)
- HW7 regrades are active on Gradescope until June 4, 11 PM
- homework 9 and solutions are available on the course website

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

$\cancel{(S_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t\geq 0}}$

$$P(\underbrace{\max_{0\leq u\leq t} B_u}_{=:S_t} > x) = P(|B_t| > x)$$

Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$. From the definition of τ_x, $\max_{0\leq u\leq t} B_u \geq x \iff \tau_x \leq t$. Then

$$P(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x) = P(\tau_x \leq t,\ B_{(t-\tau_x)+\tau_x} - \overset{=x}{B_{\tau_x}} < 0)$$

$$\overset{\text{SMP}}{=} \frac{1}{2} P(\tau_x \leq t) = \frac{1}{2} P(\max_{0\leq u\leq t} B_u \geq x)$$

Now $P(\max_{0\leq u\leq t} B_u \geq x) = P(B_t \geq x) + P(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x)$

$$\Rightarrow P(\max_{0\leq u\leq t} B_u \geq x) = 2P(B_t \geq x) = P(|B_t| \geq x) \quad \blacksquare$$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \le t \iff \max\limits_{0 \le u \le t} B_t \ge x$, so

$$P(\tau_x \le t) =$$

$$=$$

$$=$$

$\Rightarrow$ p.d.f. of τ_x $\quad f_{\tau_x}(t) =$

<u>Thm</u>. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} \int\limits_{x/\sqrt{t}}^{\infty} e^{-\frac{v^2}{2}} dv$,

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$$

Zeros of BM

Denote by $\theta(t,t+s)$ the probability that $B_u = 0$ on $(t,t+s)$

$$\theta(t,t+s) :=$$

<u>Thm</u>. For any $t, s > 0$

$$\theta(t,t+s) =$$

<u>Proof</u> Compute $P(\,B_u = 0$ for some $u \in (t,t+s]\,)$ by conditioning on the value of B_t.

$$\theta(t,t+s) =$$

(*)

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(B_u = 0 \text{ on } (t,t+s] \,|\, B_t = x) =$$

(**)

Zeros of BM

Plugging (**) into (*) gives

$$\Theta(t, t+s) = \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_0^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$+ \int_0^{\infty} P(B_u = -x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$=$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0,s]) =$

$$(*) = \int_0^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_0^s \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx =$$

Zeros of BM

$$\int_0^\infty x\, e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx =$$

$\Rightarrow$ $(*)$ $=$

Now use the change of variable $z=\sqrt{\frac{y}{t}}$, $dy = 2t\,dz$

$$(*) = \frac{\sqrt{t}}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}\,z}\cdot 2t\,dz = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{1+z^2}\,dz = \frac{2}{\pi}\arctan\left(\sqrt{\frac{s}{t}}\right)$$

$$= \frac{2}{\pi}\arccos\left(\sqrt{\frac{t}{s+t}}\right)$$

exercise

$\blacksquare$

Remark Let $T_0 := \inf\{t>0 : B_t = 0\}$. Then $P(T_0=0)=1$

There is a sequense of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros $\rightarrow$ Cantor set

Behavior of BM as $t \to \infty$

Thm. Let $(B_t)_{t\geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t\geq 0} B_t = +\infty, \inf_{t\geq 0} B_t = -\infty\right) = 1$$

(BM "oscilates with increasing amplitude")

Proof. Denote $Z = \sup_{t\geq 0} B_t$. Then for any $c > 0$

$$cZ =$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0)=p, P(Z=\infty)= 1-p$

$p=P(Z=0)$

$\Rightarrow P(Z=0)=0, P(Z=\infty)=1$. Similarly for $\inf_{t\geq 0} B_t$ ∎

Sample paths of $(B_t)_t$ are not differentiable

Thm. $P(\, B_t$ is not differentiable at zero $)=1$

Proof. $P(\sup_{t\geq 0} B_t = \infty \,, \inf_{t\geq 0} B_t = -\infty)=1.$ $(\star)$

Consider $\tilde{B}_t = t B_{1/t}$. $(\tilde{B}_t)_{t\geq 0}$ is a BM (by property (iv))

By $(\star)$, for any $\varepsilon>0$ $\exists\, t<\varepsilon,\, s<\varepsilon$ such that

$\tilde{B}_t>0,\ \tilde{B}_s<0 \Rightarrow$ only differentiable if $\tilde{B}'_0=0$

But if $\tilde{B}'_0=0$, then

for some $t>0$ and all $0<s<t$,

which imples that for all $0<s<t$, which

contradicts to $(\star)$ ∎

Thm $P((B_t)_{t\geq 0}$ is nowhere differentiable $)=1$

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$|B_t| = \begin{cases} \quad , & \text{if } B(t) \geq 0 \\ \quad , & \text{if } B(t) < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) =$

$$Var(R_t) = E(B_t^2) - \left(E(|B_t|)\right)^2 =$$

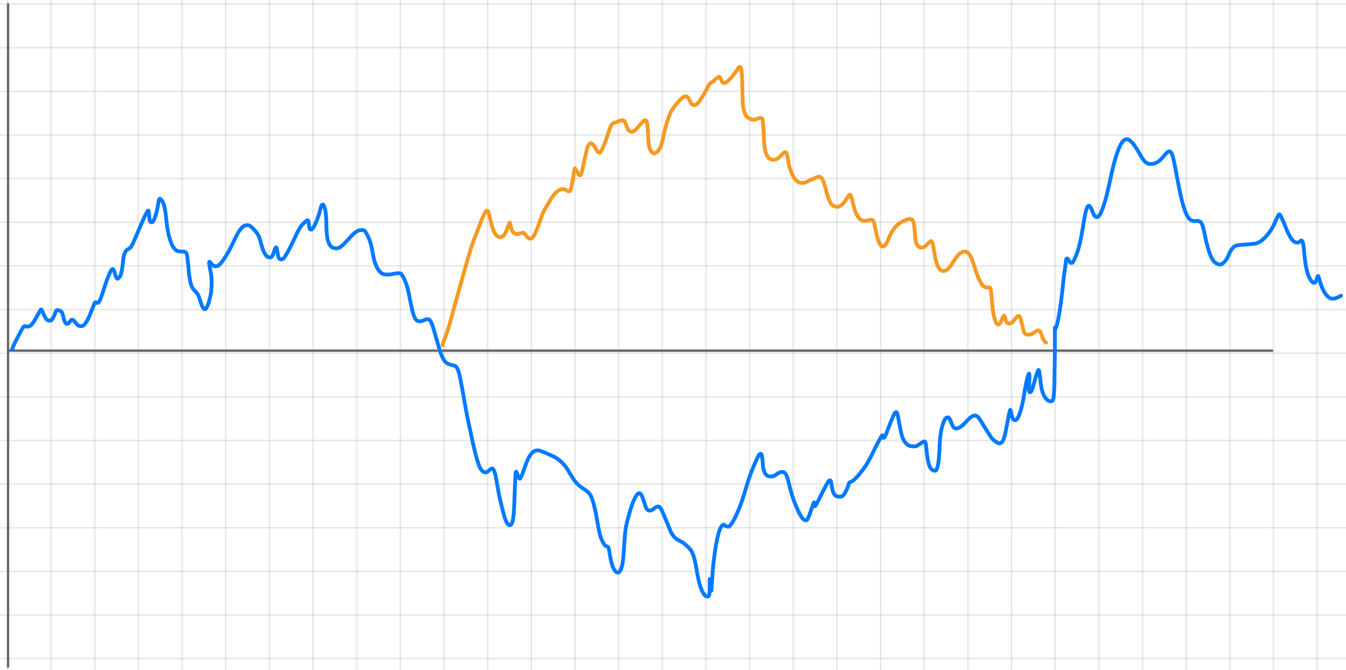
Transition density: $P(R_t \leq y \mid R_0 = x) =$

$$=$$

$$\Rightarrow p_t(x,y) =$$

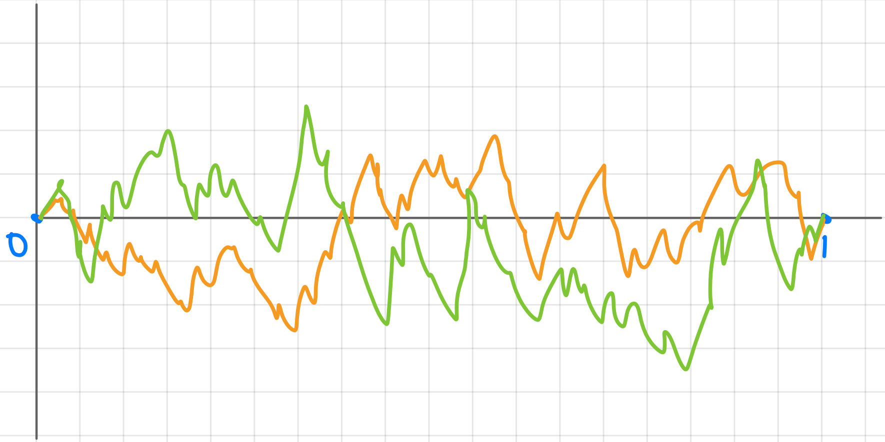
Thm (Lévy) Let $M_t = \max_{0\leq u\leq t} B_u$. Then $(M_t - B_t)_{t\geq 0}$ is a reflected BM.

Reflected BM



Brownian bridge

Brownian bridge is constructed from a BM by conditioning on the event $\{B(0)=0, B(1)=0\}$.



<u>Thm</u> 1. Brownian bridge is a continuous Gaussian process on $[0,1]$ with mean 0 and covariance function

$\Gamma(s,t) =$

Brownian motion with drift

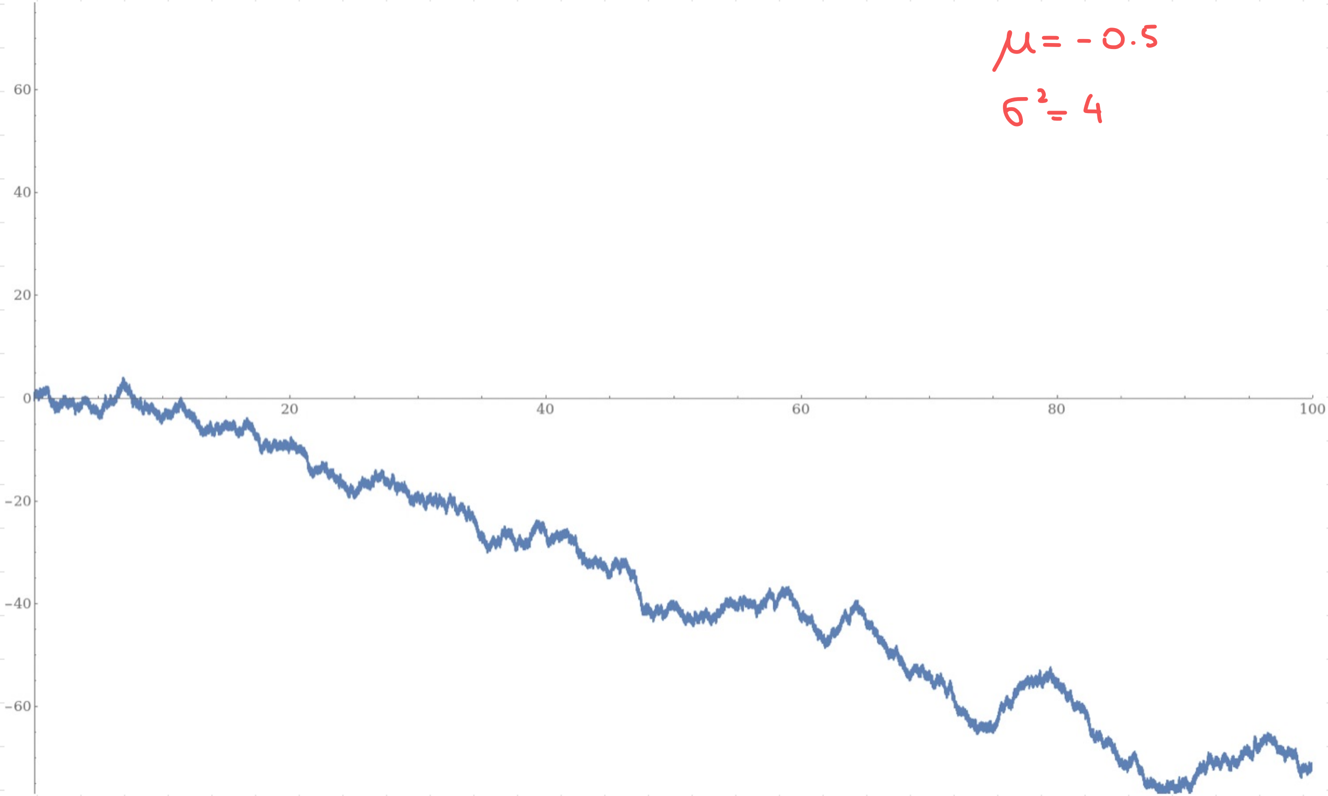
Def Let $(B_t)_{t\geq 0}$ be a standard BM. Then for $\mu \in \mathbb{R}$ and $\sigma > 0$ the process $(X_t)_{t\geq 0}$ with $X_t =$, $t \geq 0$ is called the Brownian motion with drift μ and variance paremeter σ^2.

Remark BM with drift μ and variance paremeter σ is a stochastic process $(X_t)_{t\geq 0}$ satisfying

1) $X_0 = 0$, $(X_t)_{t\geq 0}$ has continuous sample paths
2) $(X_t)_{t\geq 0}$ has independent increments
3) For $t > s$ $X_t - X_s \sim$

In particular, $X_t \sim$ $\Rightarrow X_t$ is not centered, not symmetric w.r.t. the origin

Brownian motion with drift



Gambler's ruin problem for BM with drift

Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$. Fix $a < x < b$ and denote

$$T = T_{ab} = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}, \text{ and}$$

$$u(x) = P(X_T = b \mid X_0 = x).$$

Theorem.

(i) $u(x) =$

(ii) $E(T_{ab} \mid X_0 = x) =$

No proof

Example

Fluctuations of the price of a certain share is modeled by the BM with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. You buy a share at 100 \$ and plan to sell it if its price increases to 110 \$ or drops to 95 \$.

(a) What is the probability that you will sell at profit?

(b) What is the expected time until you sell the share?

Denote by $(X_t)_{t \geq 0}$ a BM with drift $\frac{1}{10}$ and variance 4,

$x =$, $b =$, $a =$. Then $2\mu/\sigma^2 =$ and

(a) $P(X_T = 110 \mid X_0 = 100) =$

(b) $E(T \mid X_0 = 100) =$

Maximum of a BM with negative drift

Thm Let $(X_t)_{t\geq 0}$ be a BM with drift $\mu<0$, variance σ^2 and $X_0=0$. Denote $M=\max_{t\geq 0} X_t$. Then

Proof. $X_0=0$, therefore $M\geq 0$. For any $b>0$

$$P(M>b) =$$

$$=$$

$$=$$

$$P(M>b)=$$

Geometric BM

Def. Stochastic process $(Z_t)_{t\geq 0}$ is called a geometric Brownian motion with drift parameter α and variance σ^2 if $X_t =$ is a BM with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance σ^2.

In other words, $Z_t =$, where $(B_t)_{t\geq 0}$ is a standard BM and $z>0$ is the starting point $Z_0 = z$.

If $0 \leq t_1 < t_2 < \cdots < t_n$, then $\frac{Z_{t_i}}{Z_{t_{i-1}}} =$

Since B has independent increments

$\frac{Z_{t_1}}{Z_{t_0}}, \frac{Z_{t_2}}{Z_{t_1}}, \ldots, \frac{Z_{t_n}}{Z_{t_{n-1}}}$ are independent and

$\frac{Z_{t_n}}{Z_{t_0}} =$ $\leftarrow$ "relative change of price = product of independentent relative changes"

Expectation of Geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ.

Then

$E(Z_t \mid Z_0 = z) =$

$E(e^{\sigma B_t}) =$

$\Rightarrow E(Z_t \mid Z_0 = z) = z\, e^{(\alpha - \frac{1}{2}\sigma^2)t} e^{t\frac{\sigma^2}{2}} =$

Remark

It can be shown that for $0 < \alpha < \frac{1}{2}\sigma^2$ $Z_t \to 0$ as $t \to \infty$

At the same time, for $\alpha > 0$ $E(Z_t) \to \infty$.

Variance of geometric BM

$$E(Z_t^2 \mid Z_0 = z) =$$

$$=$$

$$Var(Z_t \mid Z_0 = z) =$$

Theorem.

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Then

(i) $E(Z_t \mid Z_0 = z) = z e^{\alpha t}$

(ii) $Var(Z_t \mid Z_0 = z) = z^2 e^{2\alpha t}\left(e^{\sigma^2 t} - 1\right)$

Gambler's ruin for geometric BM

Let $(Z_t)_{t\geq 0}$ be geometric BM with paremeters α and σ^2.

Let $A<1<B$, and denote $T = \min\{t : \frac{Z_t}{Z_0} = A \text{ or } \frac{Z_t}{Z_0} = B\}$.

Theorem $P\left(\frac{Z_T}{Z_0} = B\right) =$

Example Fluctuations of the price are modeled by a geometric BM with drift $\alpha = 0.1$ and variance $\sigma^2 = 4$. You buy a share at 100\$ and plan to sell it if its price increases to 110\$ or drops to 95\$.

Take $A =$, $B =$, $2\alpha/\sigma^2 =$, $1 - 2\alpha/\sigma^2 =$

$P(X_T = 110 \mid X_0 = 100) =$