MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

Today: Brownian motion

Next: PK 8.1-8.2

Week 9:

CAPES

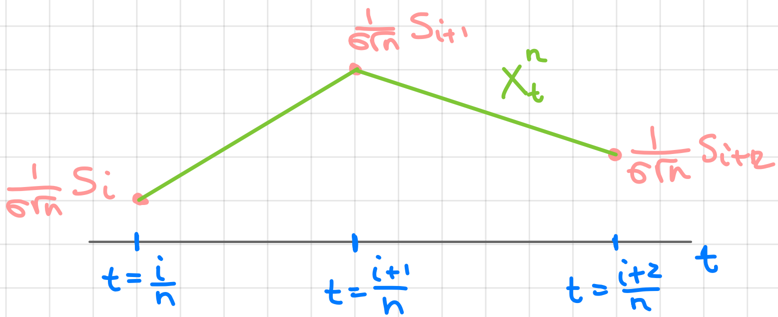
- homework 7 (due Friday, May 27)
- HW6 regrades are active on Gradescope until May 28, 11 PM
- Friday May 27 office hour: AP&M 7321

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{k=1}^{m}\xi_k$ and define

$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\,\xi_{[nt]+1}\right)$$



Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM.

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$$E(\xi_i) = 0, \quad Var(\xi_i) = 1.$$

Denote $S_m := \sum_{i=1}^{m} \xi_i$, $S_0 = 0$. $(S_m)_{m \geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$ $P(S \text{ reaches } -a \text{ before } b) = \frac{b}{a+b}$.

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}\,a \text{ before } \sqrt{n}\,b)$$

$$= \frac{\sqrt{n}\,b}{\sqrt{n}\,a + \sqrt{n}\,b} = \frac{b}{a+b}$$

$$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$$

$$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty},\ E(\tilde{\xi}_i) = 0,\ Var(\tilde{\xi}_i) = 1,\ P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|)<\infty$ $\forall t\geq 0$ and

$$E(X_t \mid \{X_u,\ 0\leq u\leq s\}) = X_s \qquad s<t$$

<u>Proposition</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) $(B_t)_{t\geq 0}$ is a martingale

(ii) $(B_t^2-t)_{t\geq 0}$ is a martingale (w.r.t. $(B_t)_{t\geq 0}$)

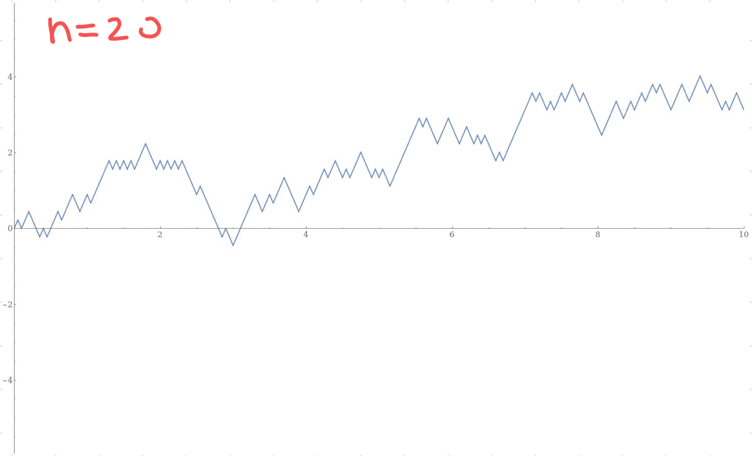
"Proof": $E(B_t \mid \{B_u, 0\leq u\leq s\}) = E\left(B_s + (B_t - B_s) \mid \{B_u, 0\leq u\leq s\}\right) = B_s + 0$

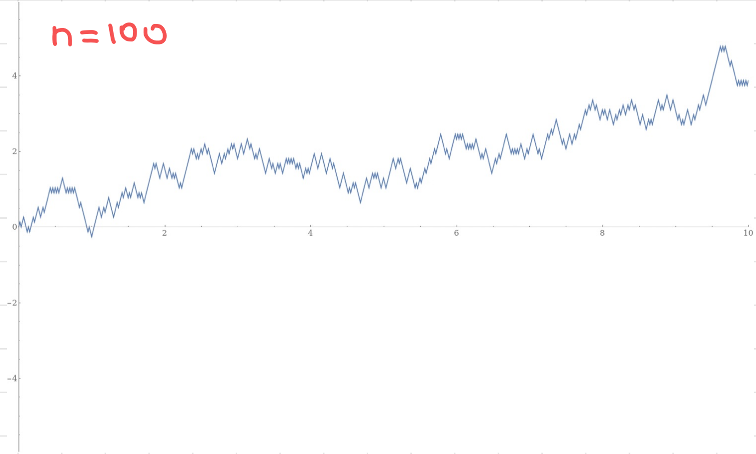
$$E\left(B_t^2 - t \mid \{B_u, 0\leq u\leq s\}\right) = E\left(B_s^2 + 2B_s(B_t-B_s) + (B_t-B_s)^2 - t \mid \{B_u, 0\leq u\leq s\}\right)$$

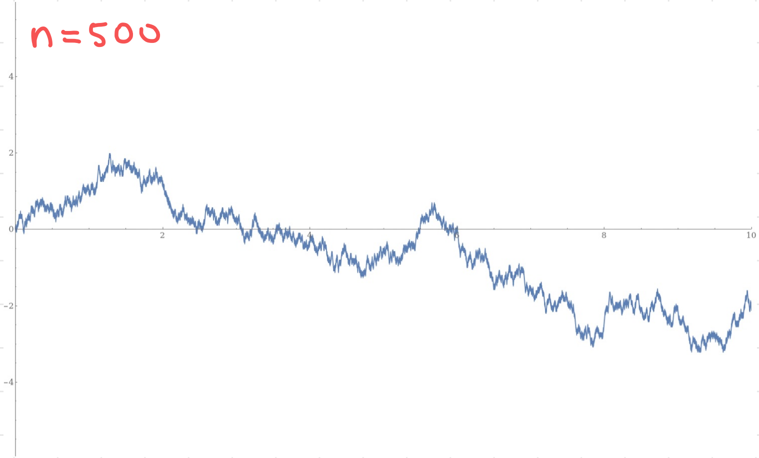
$$= B_s^2 + 0 + t - s - t = B_s^2 - s$$

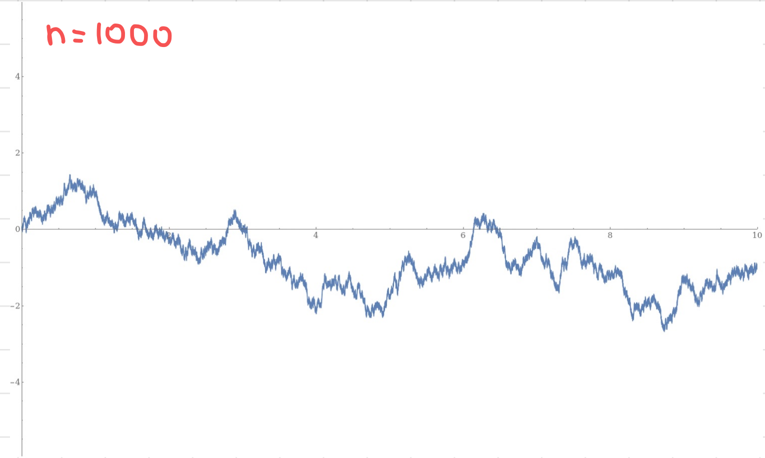
<u>Thm</u> (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2-t)_{t\geq 0}$ is a martingale. Then $(X_t)_{t\geq 0}$ is a BM.

Approximating a BM with random walks X_t^n









Stopping times and the strong Markov property (lec. ?)

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a stopping time if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec. ?)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

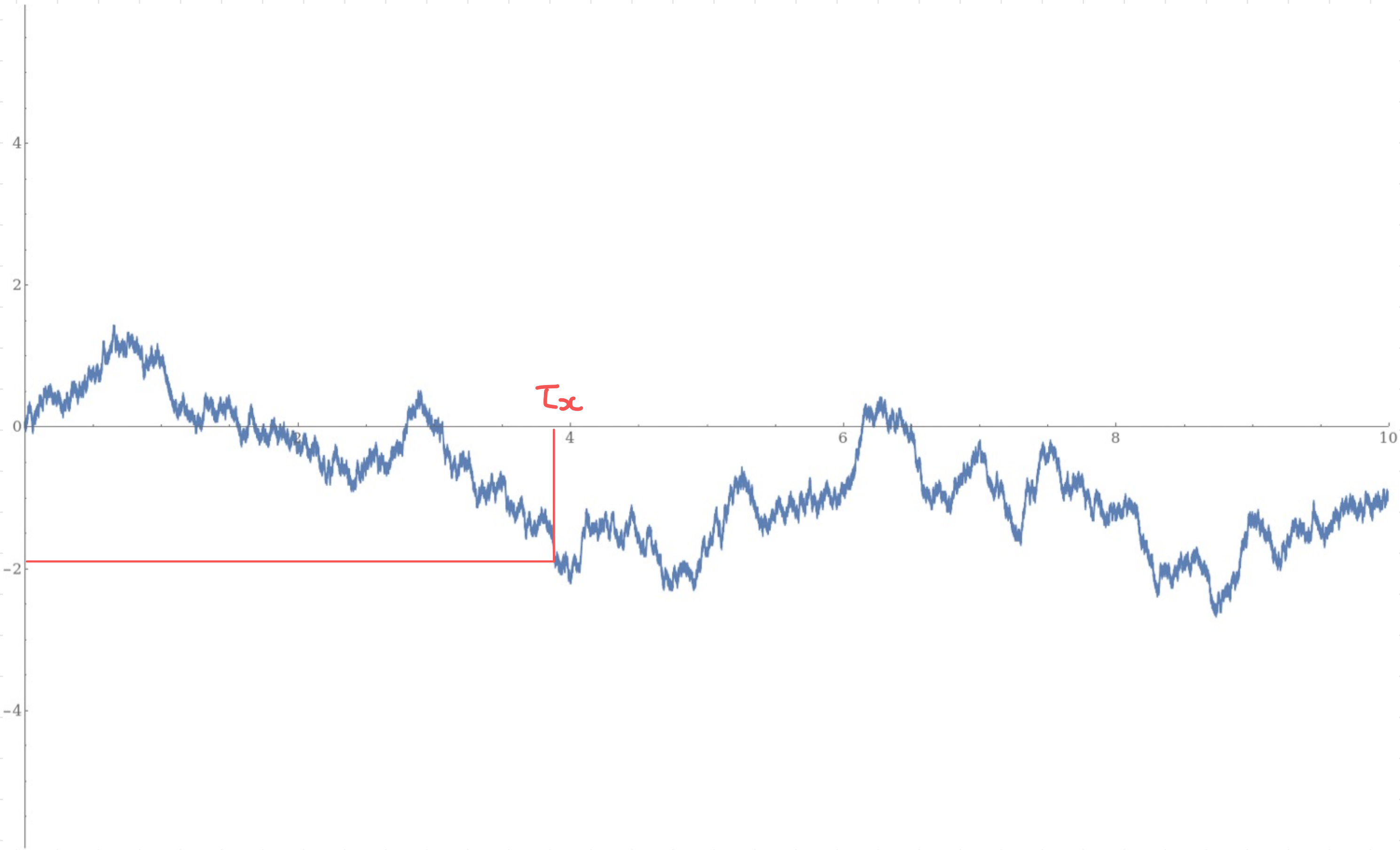
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$$\tau_x = \min\{t : B_t = x\}.$$ Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from 0
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)



4
2
0
-2
-4
2
4
6
8
10
τ_x

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$ ~~$(S_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t\geq 0}$~~

$$P\Big(\underbrace{\max_{0\leq u\leq t} B_u}_{= S_t} > x\Big) = P(|B_t| > x) = 2\,P(B_t > x)$$

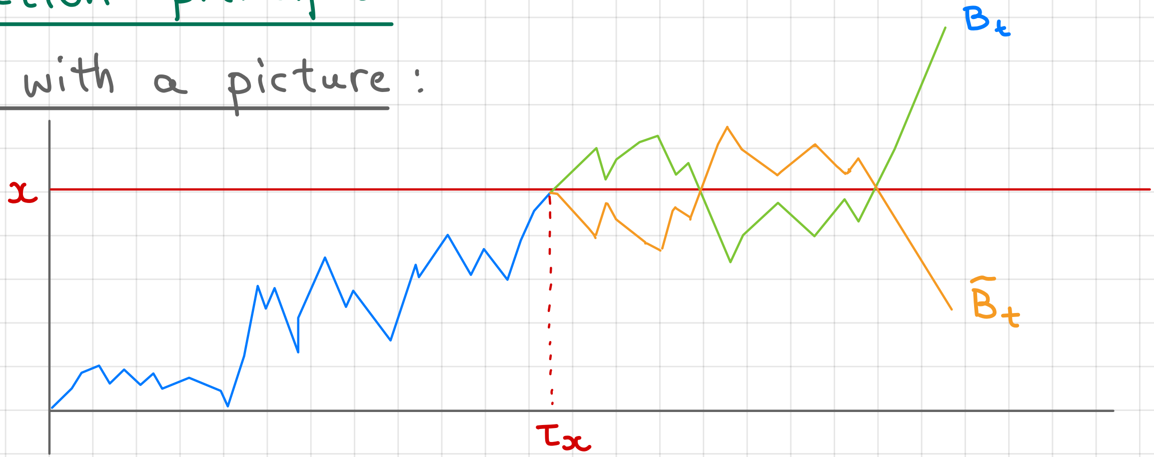
Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$. From the definition of τ_x, $\max_{0\leq u\leq t} B_u \geq x \iff \tau_x \leq t$. Then

$$P\left(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x\right) =$$

Now $P\left(\max_{0\leq u\leq t} B_u \geq x\right) =$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t, & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}), & t > \tau_x \end{cases}$$

$\Rightarrow$ to each sample path with $\max\limits_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max\limits_{0\leq u\leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P\Big(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x\Big) = \underset{\overset{\shortparallel}{P\left(\max\limits_{0\leq u\leq t} \geq x,\ B_t > x\right)}}{P(\tilde{B}_t > x)} \Rightarrow P\Big(\max_{0\leq u\leq t} B_u \geq x\Big) = 2P(B_t \geq x) \quad \blacksquare$$