# MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

**Today: Brownian motion** 

# Next: PK 8.1-8.2

Week 9:

homework 7 (due Friday, May 27)

HW6 regrades are active on Gradescope until May 28, 11 PM

### Brownian motion. History

Critical observation: Robert Brown (1827), botanist,

movement of pollen grains in water

- First (?) mathematical analysis of Brownian motion:
  Louis Bachelier (1900), modeling stock market
  fluctuations
- · Brownian motion in physics : Albert Einstein (1905) and

Marian Smoluchowski (1906), explained the

phenomenon observed by Brown

· First rigorous construction of mathematical Brownian

motion: Norbert Wiener (1923)

Brownian motion = Wiener process in mathematics

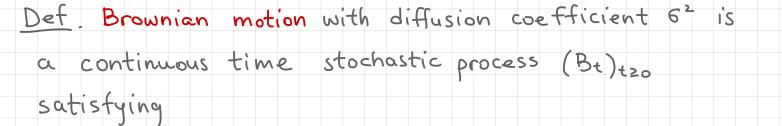
#### Brownian motion. Motivation

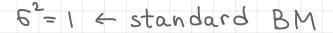
almost all interesting classes of stochastic processes

contain Brownian motion : BM is a

- martingale
- Markov process
- Gaussian process
- Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for
  - more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

#### Brownian motion. Definition

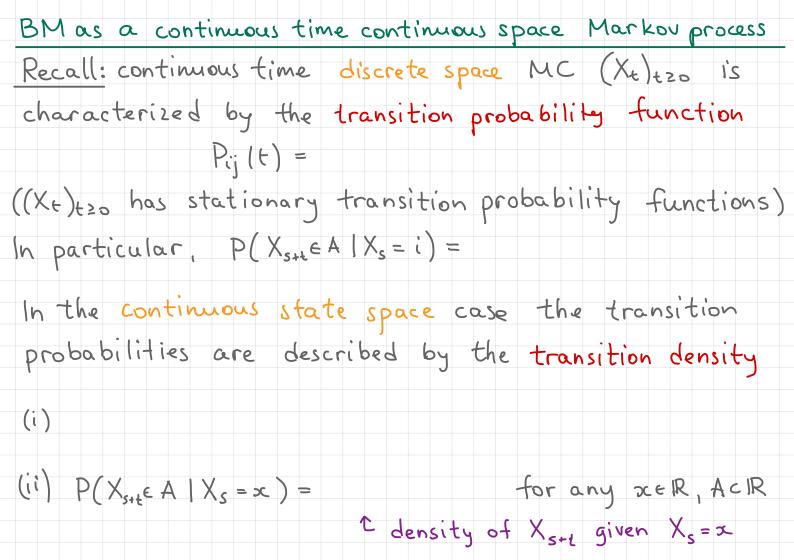




(i)

(ii)

(iii)



BM as a continuous time continuous space Markov process

Then  $(B_t)_{t\geq 0}$  is a with transition

density

Informal explanation: Independent stationary increments imply that  $(B_t)_{t\geq 0}$  is Markov with stationary transition density. Given  $B_s = x$ , information before time s is irrelevant.

$$P(B_{s+t} \leq u | B_s = x) =$$

BM as a continuous time continuous space Markov process Let t, ctz c... et , co, (ai, bi) c IR. Then  $P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$ 2 -\* More generally,  $P(B_{t_1}e(a_1,b_1), B_{t_2}e(a_2,b_2), \dots, B_{t_n}e(a_n,b_n))$ =  $\int P_{t_1}(o_1 x_1) P_{t_2-t_1}(x_1, x_2) \cdots P_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_1 \cdots dx_n$ (a, b) x ... x (an, bn)

(iii) density Pt satisfies [K forward] t diffusion equation

(ii) density pt satisfies [K backward]

We call  $(P_t)_{t_{20}}$  the transition semigroup  $[P_{s,t} f(x) = P_s(P_t f(x))]$ <u>Proposition</u> Let  $(P_t)_{t_{20}}$  be the transition semigroup of BM. Then (i) the "infinitesimal generator" of P(t) is given by

evolves in time :

Let  $(X_t)_{t \ge 0}$  be a Markov process. Suppose we want to know how the distribution of  $X_t$ 

Diffusion equation. Transition semigroup. Generator

## BM as a Gaussian process

<u>Def</u> Stochastic process (Xt)tzo is called a Gaussian process if for any Oft, <t2 <... < tn

(X<sub>t</sub>,..., X<sub>t</sub>n) is a Gaussian vector, or equivalently for any C<sub>1</sub>,..., Cn ∈ IR

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is

uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t)$$
 and  $\Gamma(s,t) = Cov(X_s,X_t) \ge 0$   
t covariance function

### BM as a Gaussian process

Proposition BM is a Gaussian process with

and

Proof. For any Ost, <tz <-- < tn, Bt; -Bt;, are indep.

Gaussian, thus n Z Ci Bti= is also Gaussian.

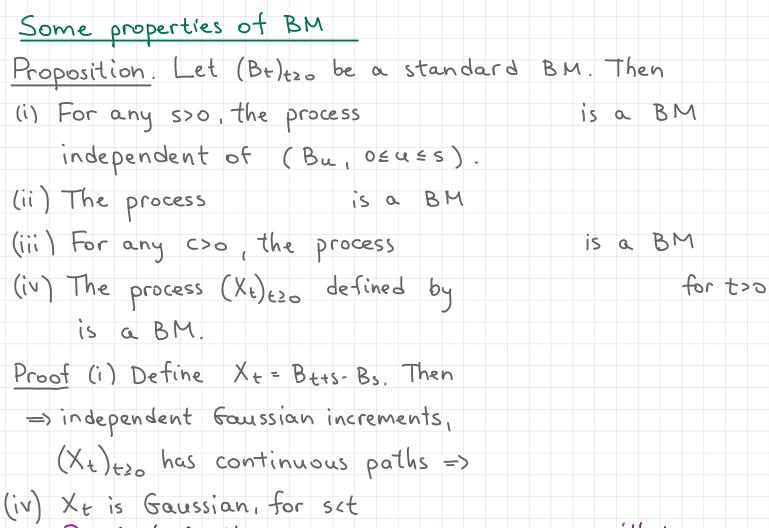
By definition

Then  $\Gamma(s,t)=$ 

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. Let set.



Proof of lim Xt = 0 is more technical, thus omitted.