# MATH180C: Introduction to Stochastic Processes II

Lecture A00: math.ucsd.edu/~ynemish/teaching/180cA Lecture B00: math.ucsd.edu/~ynemish/teaching/180cB

**Today: Brownian motion** 

# Next: PK 8.1-8.2

Week 9:

homework 7 (due Friday, May 27)

HW6 regrades are active on Gradescope until May 28, 11 PM

### Brownian motion. History

Critical observation: Robert Brown (1827), botanist,

movement of pollen grains in water

- First (?) mathematical analysis of Brownian motion:
  Louis Bachelier (1900), modeling stock market
  fluctuations
- · Brownian motion in physics : Albert Einstein (1905) and

Marian Smoluchowski (1906), explained the

phenomenon observed by Brown

· First rigorous construction of mathematical Brownian

motion: Norbert Wiener (1923)

Brownian motion = Wiener process in mathematics

#### Brownian motion. Motivation

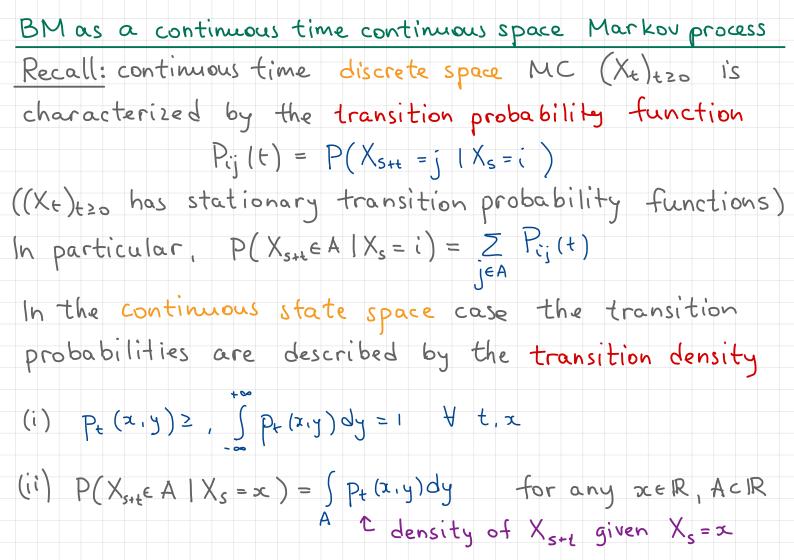
almost all interesting classes of stochastic processes

contain Brownian motion : BM is a

- martingale
- Markov process
- Gaussian process
- Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for
  - more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

#### Brownian motion. Definition

Def Brownian motion with diffusion coefficient 62 is a continuous time stochastic process (Bt) t20 satisfying (i) B(o)=0, B(t) is continuous as a function of t (ii) For all Osseter B(+)-B(s) is a Gaussian r.v. with mean 0 and variance  $\sigma^2(t-s)$ (iii) The increments of B are independent : if ofto <t, <- <tn then { Bti - Bti, j; are independent Gaussian r.v.s 5<sup>2</sup>=1 < standard BM



BM as a continuous time continuous space Markov process

Propotition. Let  $(B_t)_{t\geq 0}$  be a standard BM. Then  $(B_t)_{t\geq 0}$  is a Markov process with transition density  $\frac{1}{2\pi t} = \frac{1}{2\pi t} (y-x)^2$ 

Informal explanation: Independent stationary increments imply that (Bt) == is Markov with stationary transition density. Given Bs=x, Bs+t = Bs + Bs+t - Bs ~ N(x,t) information before time s is irrelevant.  $P(B_{s+t} \leq u \mid B_s = x) = P(B_s + (B_{s+t} - B_s) \leq u \mid B_s = z)$  $= P(x + B_{t+s} - B_s \le u) = \int_{(2\pi t)}^{u} \frac{1}{(2\pi t)} e^{-\frac{(y-x)^2}{2t}} dy$ 

BM as a continuous time continuous space Markov process

Let tictz c... ctn co, (ai, bi) c.R. Then

 $P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$ 

 $= \int_{a_{1}} P(B_{t_{1}} \in (a_{1}, b_{1}), B_{t_{2}} \in (a_{2}, b_{2}) | B_{t_{1}} = x_{1}) P_{t_{1}}(o_{1}x_{1}) dx_{1}$ =  $\int_{b_{1}} P(B_{t_{2}} \in (a_{2}, b_{2}) | B_{t_{1}} = x_{1}) P_{t_{1}}(o_{1}x_{1}) dx_{1}$ 

=  $\int \int P_{t_1}(o, x_1) P_{t_2-t_1}(x_1, x_2) dx_1 dx_2$ 

More generally, $P(B_{t_{i}}e(a, b_{i}), B_{t_{2}}e(a_{2}, b_{2}), ..., B_{t_{n}}e(a_{n}, b_{n})) = \int ... \int P_{t_{1}}(o_{1}x_{1}) P_{t_{2}-t_{1}}(x_{1}, x_{2}) \cdots P_{t_{n}-t_{n-1}}(x_{n-1}, x_{n}) dx_{1} \cdots dx_{n}$  $(a, b_{i})x \cdots x(a_{n}, b_{n})$  Diffusion equation. Transition semigroup. Generator

Let (Xt)tzo be a Markov process,

Suppose we want to know how the distribution of Xt

evolves in time:

 $E(f(X_{stt})|X_{s}=x) = \int f(y) p_{t}(x,y) dy = P_{t}f(x)$ 

We call  $(P_t)_{t_{20}}$  the transition semigroup  $[P_{s,t}f(x)=P_s(P_tf(x))]$ <u>Proposition</u> Let  $(P_t)_{t_{20}}$  be the transition semigroup of BM. Then (i) the infinitesimal generator of P(t) is given by

 $Qf(z) = \frac{1}{2}\frac{d}{dx^2}f(z)$ 

(ii) density  $p_t$  satisfies  $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x,y)$  [K backward]

(iii) density Pt satisfies  $\frac{\partial}{\partial t} Pt(x,y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} Pt(x,y) [K forward] t diffusion equation$ 

## BM as a Gaussian process

<u>Def</u> Stochastic process (Xt)tzo is called a Gaussian process if for any Oft, <t2 <... < tn

(X<sub>t</sub>,..., X<sub>t</sub>n) is a Gaussian vector, or equivalently for any C<sub>1</sub>,..., Cn ∈ IR

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is

uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t)$$
 and  $\Gamma(s,t) = Cov(X_s,X_t) \ge 0$   
t covariance function

### BM as a Gaussian process

Proposition BM is a Gaussian process with

and

Proof. For any Ost, <tz <-- < tn, Bt; -Bt;, are indep.

Gaussian, thus n Z Ci Bti= is also Gaussian.

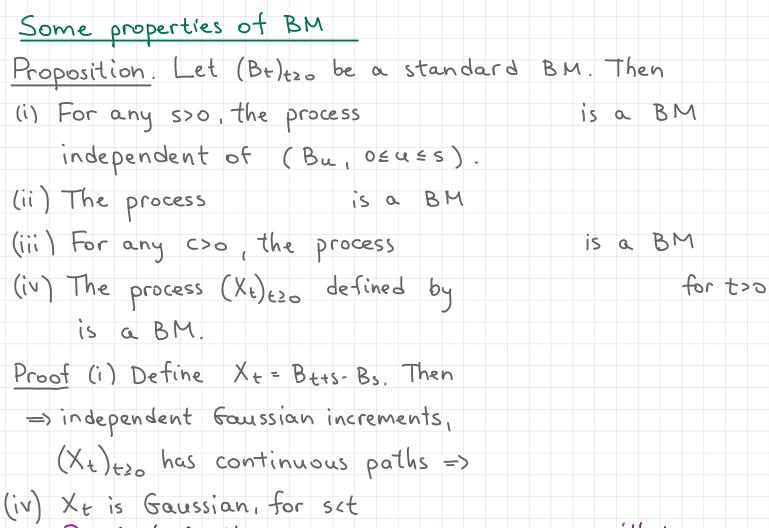
By definition

Then  $\Gamma(s,t)=$ 

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. Let set.



Proof of lim Xt = 0 is more technical, thus omitted.