MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

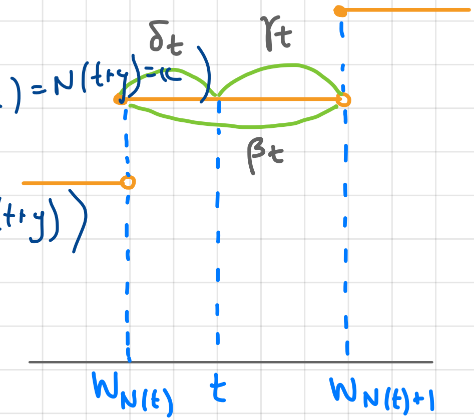
- homework 5 (due Friday, May 6)
- regrades for Midterm 1 and HW4 active until May 7, 11PM

Joint distribution of age and excess life

From the definition of γ_t and δ_t

$$P(\delta_t \geq x, \gamma_t > y) \qquad (x \leq t)$$

$$= \sum P(N(t-x) = N(t+y) = k)$$



$$= P(W_{N(t)} \leq t-x, W_{N(t)+1} > t+y) = P(N(t-x) = N(t+y))$$

- Partition w.r.t. the values of $N(t)$

$$= \sum_{k=0}^{\infty} P(W_k \leq t-x, W_{k+1} > t+y)$$

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

$$= \overbrace{1 - F(t+y)}^{P(W_1 > t+y)} + \sum_{k=1}^{\infty} \int_0^{\infty} P(W_k \leq t-x, \overset{W_k + X_{k+1}}{W_{k+1}} > t+y \mid W_k = u)\, dF^{*k}(u)$$

$$= P(X_{k+1} > t+y-u) = 1 - P(X_{k+1} \leq t+y-u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} P(u + X_{k+1} > t+y)\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, d\underbrace{\sum_{k=1}^{\infty} F^{*k}(u)}_{M(u)}$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(\underbrace{t+y-u}_{=:w}))\, m(u)\, du$$

$$= 1 - F(t+y) + \int_{y+x}^{y+t} (1 - F(w))\, m(t+y-w)\, dw$$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$$\lim_{t\to\infty} P(\delta_t \geq x, \gamma_t > y) = \lim_{t\to\infty} \left[1 - F(t+y) + \int_{y+x}^{y+t} (1 - F(w)) \left\{\frac{1}{\mu} + \varepsilon(t+y-w)\right\} dw\right]$$

$$= \int_{y+x}^{\infty} (1 - F(w)) \frac{1}{\mu}\, dw + \underbrace{\lim_{t\to\infty} \int_{y+x}^{y+t} (1 - F(w))\, \varepsilon(t+y-w)\, dw}_{\to 0}$$

Exercise

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$
$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, dM(u)$$

(b) if additionally the interrenewal times are continuous,
$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w))\, dw \qquad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) = \frac{1}{\mu}(1 - F(x)) \qquad \left\{ \quad \frac{1}{E(X)} \underbrace{\int_0^{\infty} P(X > x)\, dx}_{= E(X)} \right.$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on [0,1] (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

$$\lim_{t\to\infty} P\left(\gamma_t \le \frac{1}{2}\right) = \int_0^{1/2} 2\cdot(1-x)\,dx = 1 - x^2 \Big|_0^{1/2} = 0.75$$

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

$$\lim_{t\to\infty} P\left(\delta_t \le \frac{1}{2}\right) = \int_0^{1/2} 2\cdot(1-x)\,dx = 0.75$$