MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- homework 5 (due Friday, May 6)
- regrades for Midterm 1 and HW4 active until May 7, 11PM

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0,\infty)$.

Thm.

$$P\left(\lim_{t\to\infty} N(t) = +\infty\right) = 1$$

Proof. $N(t)$ is nondecreasing, therefore $\exists \lim_{t\to\infty} N(t) =: N_\infty$ $\in (0,+\infty) \cup \{+\infty\}$

N_∞ is the total number of events ever happened.

$N_\infty \leq k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $1 \leq i \leq k+1$

$$P(N_\infty < \infty) = P(X_i = \infty \text{ for some } i) \leq \sum_{i=1}^{\infty} \underbrace{P(X_i = \infty)}_{=0} = 0$$

Thm (Pointwise renewal thm).

$$P\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right) = 1 \qquad \mu = E(X_i)$$

Elementary Renewal Theorem

<u>Thm</u>. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^K x f(x)\,dx \le K \int_0^K f(x)\,dx = K$$

<u>Proof</u>. (Only for bounded X_i: $\exists\, K$ s.t. $P(X_1 \le K) = 1$)

First note that $W_{N(t)+1} = t + \gamma_t$

In lecture 13 we showed that $E(W_{N(t)+1}) = \mu(M(t)+1)$,

so $$M(t) = \frac{t + E(\gamma_t)}{\mu} - 1$$

$$\frac{M(t)}{t} = \frac{1}{\mu} + \frac{1}{t}\left(\frac{E(\gamma_t)}{\mu} - 1\right) \longrightarrow \frac{1}{\mu} \quad \text{as } t \to \infty$$

If $X_i \le K$, then $\gamma_t \le K \Rightarrow E(\gamma_t) \le K$ ■

Ex. $(X_n)_{n\ge 0}$: 1) $P(\lim_{n\to\infty} X_n = 0) = 1$ 2) $\lim_{n\to\infty} E(X_n) > 0$

Asymptotic distribution of N(t)

Thm. Let $N(t)$ be a renewal process with $E(X_1)=\mu$ and $Var(X_1)=\sigma^2$, then

1)
$$\lim_{t\to\infty} \frac{Var(N(t))}{t} = \frac{\sigma^2}{\mu^3}$$

2)
$$\lim_{t\to\infty} P\left(\frac{N(t)-E(N(t))}{\sqrt{Var(N(t))}} \le x\right) = \lim_{t\to\infty} P\left(\frac{N(t)-\frac{t}{\mu}}{\sqrt{\frac{\sigma^2}{\mu^3}t}} \le x\right)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

$N(t) \approx \frac{t}{\mu} + \sqrt{\frac{\sigma^2}{\mu^3}t}\cdot Z$, where $Z\sim N(0,1)$ for t large

No proof.

Elementary renewal theorem and continuous X_i's

Two more results (without proofs) about the limiting behaviour of $M(t)$ for models with continuous interrenewal times.

<u>Thm</u>. Let $E(X_1) = \mu$ and let $m(t) = \frac{d}{dt} M(t)$ be the renewal density. Then

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \frac{dM(t)}{dt} = \frac{1}{\mu}$$

<u>Remark</u> $\lim_{t \to \infty} \frac{f(t)}{t} = \alpha$ does not imply in general $\lim_{t \to \infty} f'(t) = \alpha$

(E.g., take $f(t) = t + \sin(t)$)

<u>Thm</u>. If additionally $Var(X_1) = \sigma^2$, then

$$\lim_{t \to \infty} \left(M(t) - \frac{t}{\mu} \right) = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

Example: $X_i \sim \text{Gamma}(2,1)$

Let $N(t)$ be a renewal process with interrenewal times X_i having Gamma distribution with parameters $(2,1)$ i.e., $f_{X_1}(t) = t\,e^{-t}$. Then from the properties of the Gamma distribution (or from direct computations)

$$X_1 + \cdots + X_n \sim \text{Gamma}(2n, 1), \text{ so}$$

$$f^{*n}(t) = \frac{t^{2n-1}}{(2n-1)!} e^{-t}, \quad \text{for } t > 0$$

We can compute the renewal density

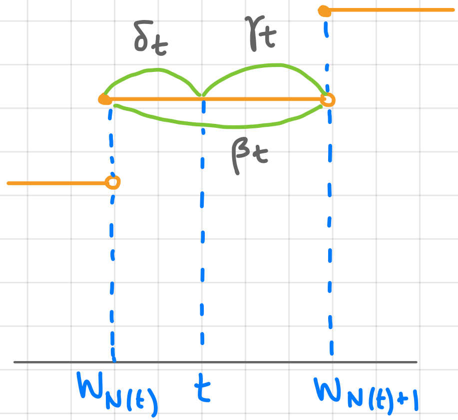
$$m(t) = \sum_{n=1}^{\infty} f^{*n}(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} e^{-t} = \frac{e^{t} - e^{-t}}{2} e^{-t} = \frac{1 - e^{-2t}}{2}$$

so that $M(t) = \int_0^t m(x)\,dx = \frac{t}{2} - \frac{1}{4} + \frac{1}{4} e^{-2t}$

Finally, $E(X_1) = \mu = 2$, $Var(X_1) = \sigma^2 = 2$, so $\frac{\sigma^2 - \mu^2}{2\mu^2} = -\frac{2}{2 \cdot 4} = \frac{-1}{4}$

Joint distribution of age and excess life

From the definition of γ_t and δ_t



$P(\delta_t \geq x, \gamma_t > y)$ $\quad (x \leq t)$

$=$

- Partition wrt the values of $N(t)$

$=$

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

$=$

$=$

$=$

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$P(\delta_t \geq x, \gamma_t > y) =$

$=$

$=$

$=$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$\lim_{t \to \infty} P(\delta_t \geq x, \gamma_t > y) =$

$=$

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, dM(u)$$

(b) if additionally the interrenewal times are continuous,

$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w))\, dw \qquad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) =$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on $[0,1]$ (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\,dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\,dx$

<u>Def</u>. Function h is called locally bounded if

<u>Def</u>. Function h is absolutely integrable if

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies , then H is locally bounded

and

(b) Conversely, if H is a locally bounded solution to (*),

then

[convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable , then

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to (*) given by (**)

Examples

- Renewal function: $M(t)$ satisfies

  and

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

  and

  (in the Riemann-Stieltjes sense)

  f is absolutely integrable, , so

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W'
- W'

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then ; if $X_1 < t$ condition on $X_1 = s$

$E(\gamma_t) =$

$E(\gamma_t \mathbb{1}_{X_1 \le t}) =$

$=$

$=$

$=$

$=$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E((X_1-t)\mathbb{1}_{X_1>t}) =$$

$$=$$

Since we assume that $E(X_1)=\sigma^2$,

and

Finally, we have that

$$H(t) =$$

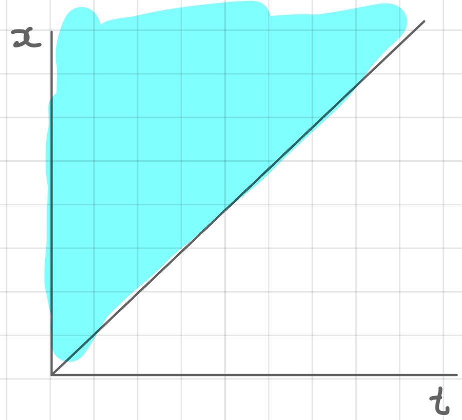
therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt =$$



$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) =$$

Similarly $\lim_{t\to\infty} E(\delta_t) =$, $\lim_{t\to\infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

therefore, $\lim_{t \to \infty} E(\gamma_t) =$

And the long run expected time between two consecutive earthquakes is