MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Renewal processes
Poisson process as a renewal process
Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- homework 4 (due Friday, April 29)
- regrades for HW 3 active until April 30, 11PM

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \ldots$ and renewal times $(W_n)_{n \geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\, E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) =$

$E(X_2 + \cdots + X_{N(t)+1}) =$

$=$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) =$$

$$=$$

Since $N(t) \geq j-1 \iff W_{j-1} \leq t \iff X_1 + X_2 + \cdots + X_{j-1} \leq t$

$$=$$

$$=$$

$$=$$

Remark For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

Proposition 3. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

Proof. We showed in Proposition 1 that

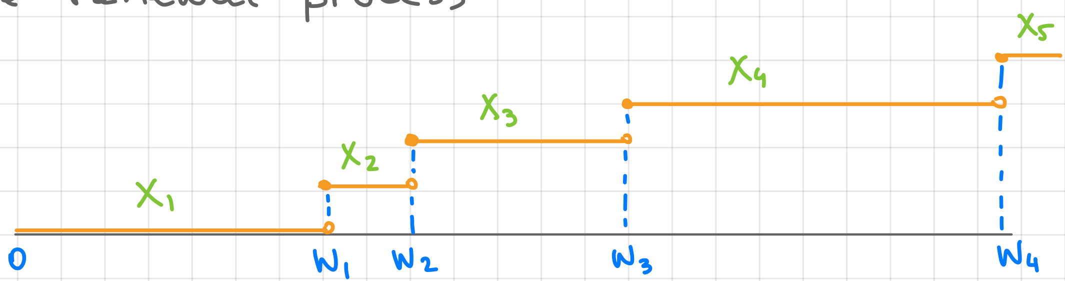
$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then $M * F =$

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

<u>Example</u> Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$F_2(t) =$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) =$$

$F * F(t) =$

$F^{*3}(t) =$

$\vdots$

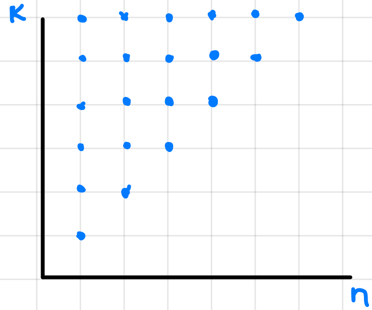
$F^{*n}(t) =$

Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) =$$

$$=$$

$$=$$



$$M(t) =$$

Renewal density

Proposition Let $N(t)$ be a renewal process with continuous interrenewal times X_i having density $f(x)$. Denote $m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then

and (*)

renewal density

Proof: $\frac{d}{dt} F^{*n}(t) =$ ■

Example: Compute the renewal density for PP using (*).

$f(x) = \lambda e^{-\lambda x}$, so (*) becomes

$m(t) =$

$=$

(cont.)

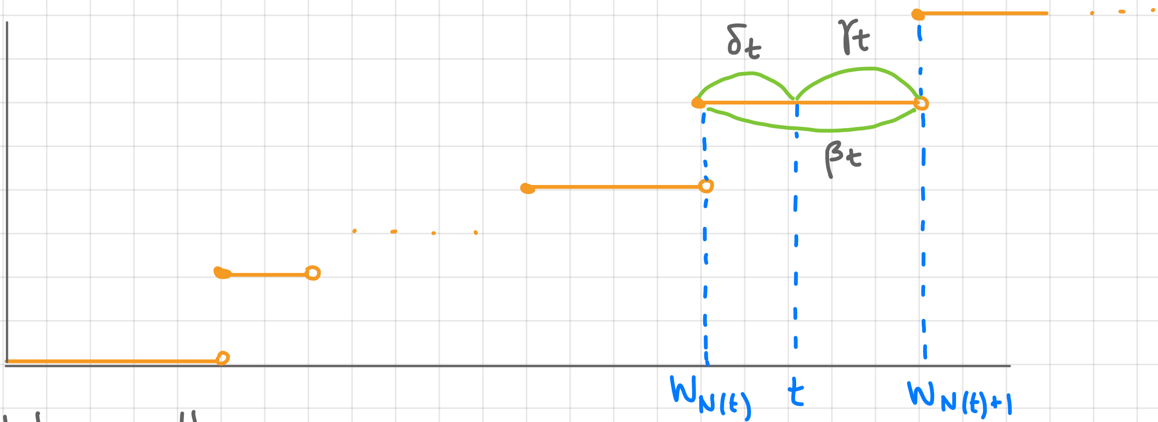
$e^{\lambda t} m(t) =$ $\leftarrow$ differentiate

$\Rightarrow$

Indeed,

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life

$$P(\gamma_t > x) =$$

- current life δ_t

$$P(\delta_t > x) = \begin{cases} & \\ & \end{cases}$$

- total life $\beta_t = \gamma_t + \delta_t$

$$E(\gamma_t + \delta_t) =$$

$$=$$

Excess life and current life of PP (cont.)

- Joint distribution of (γ_t, δ_t)

$$P(\gamma_t > x, \delta_t > y) = \begin{cases} & \\ & \end{cases}$$

$\Longrightarrow$