MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Renewal processes
Poisson process as a renewal process
Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- homework 4 (due Friday, April 29)
- regrades for HW 3 active until April 30, 11PM

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\, E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) = E(X_1 + X_2 + \dots + X_{N(t)+1}) = \underbrace{E(X_1)}_{=\mu} + E(X_2 + \dots + X_{N(t)+1})$

$$E(X_2 + \dots + X_{N(t)+1}) = E(X_2 \mid N(t)=1)\, P(N(t)=1)$$
$$+ E(X_2 + X_3 \mid N(t)=2)\, P(N(t)=2)$$
$$+ E(X_2 + X_3 + X_4 \mid N(t)=3)\, P(N(t)=3)$$
$$\vdots$$
$$+ E\left(\sum_{k=2}^{n+1} X_k \mid N(t)=n\right) P(N(t)=n) + \dots$$

$$= \sum_{n=1}^{\infty} E(X_2 \mid N(t)=n)\, P(N(t)=n) + \sum_{n=2}^{\infty} E(X_3 \mid N(t)=n)\, P(N(t)=n) + \dots$$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) = \sum_{j=2}^{\infty}\sum_{n=j-1}^{\infty} E(X_j \mid N(t)=n)\,P(N(t)=n)$$

$$= \sum_{j=2}^{\infty} E(X_j \mid N(t)\geq j-1)\,P(N(t)\geq j-1)$$

Since $N(t)\geq j-1 \iff W_{j-1}\leq t \iff X_1+X_2+\dots+X_{j-1}\leq t$

$$= \sum_{j=2}^{\infty} E(\underbrace{X_j \mid X_1+X_2+\dots+X_{j-1}\leq t}_{\text{independent}})\,P(N(t)\geq j-1)$$

$$= \sum_{j=2}^{\infty} E(X_j)\,P(N(t)\geq \overbrace{j-1}^{=\ell}) = \mu\cdot\sum_{\ell=1}^{\infty} P(N(t)\geq \ell)$$

$$= \mu\,E(N(t)) = \mu\cdot M(t) \qquad \blacksquare$$

<u>Remark</u> For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

<u>Proposition 3.</u> Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

$$M(t) = F(t) + M * F(t) = F(t) + \int_0^t M(t-x)\, dF(x)$$

renewal equation

<u>Proof</u>. We showed in Proposition 1 that

$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then

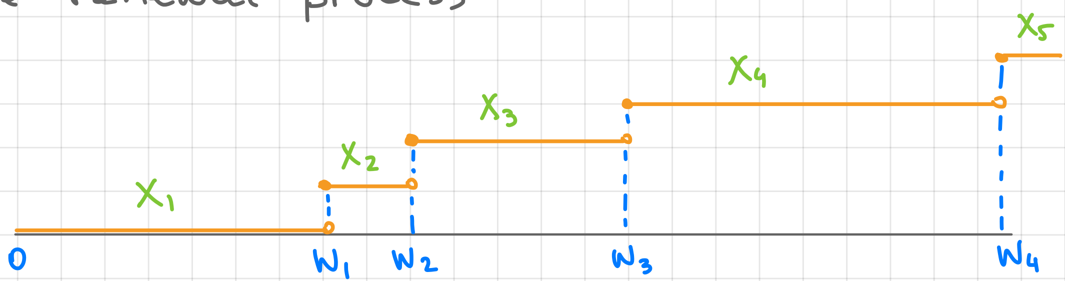
$$M * F = \left(\sum_{n=1}^{\infty} F^{*n}\right) * F = \sum_{n=2}^{\infty} F^{*n} = \sum_{n=1}^{\infty} F^{*n} - F$$

$$= M - F \qquad \blacksquare$$

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$$F_2(t) = \underbrace{\int_0^t \left(1 - e^{-\lambda(t-x)}\right) \lambda e^{-\lambda x} \, dx}_{\int F(t-x)\, dF(x)} = 1 - e^{-\lambda t} - \lambda \int_0^t e^{-\lambda t} \, dx = F(t) - \overbrace{\lambda t e^{-\lambda t}}^{\varphi_1(t)}$$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) = \int_0^t \overbrace{\frac{\lambda^k (t-x)^k}{k!} e^{-\lambda(t-x)}}^{\varphi_k(t-x)} \lambda e^{-\lambda x} \, dx = \frac{(\lambda t)^{k+1}}{(k+1)!} e^{-\lambda t} = \varphi_{k+1}$$

$$F * F(t) = F(t) - \varphi_1(t)$$

$$F^{*3}(t) = (F - \varphi_1) * F = F * F - \varphi_1 * F = F - \varphi_1 - \varphi_2$$

$$\vdots$$

$$F^{*n}(t) = F - \varphi_1 - \varphi_2 \cdots - \varphi_{n-1}$$

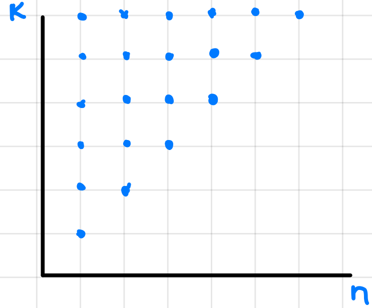
Poisson process as a renewal process (cont.)

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}$$

$$F = 1 - \varphi_0$$

$$\sum_{n=1}^{\infty} F^{*n}(t) = \sum_{n=1}^{\infty} \left(1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \right) = e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t} \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}$$



$$= \lambda t \, e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} = \lambda t$$

$$M(t) = \lambda t$$