MATH180C: Introduction to Stochastic Processes II

Lecture A00: math-old.ucsd.edu/~ynemish/teaching/180cA
Lecture B00: math-old.ucsd.edu/~ynemish/teaching/180cB

Today: Renewal processes

Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- homework 4 (due Friday, April 29)
- regrades for HW 3 active until April 30, 11PM

Renewal process. Definition

Def. Let $\{X_i\}_{i\geq 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \cdots + X_n$, $n \geq 1$, and $W_0 := 0$.

We call the counting process

$$N(t) = \#\{k : W_k \leq t\} = \max\{n : W_n \leq t\}$$

the renewal process.

Remarks. 1) W_n are called the waiting/renewall times

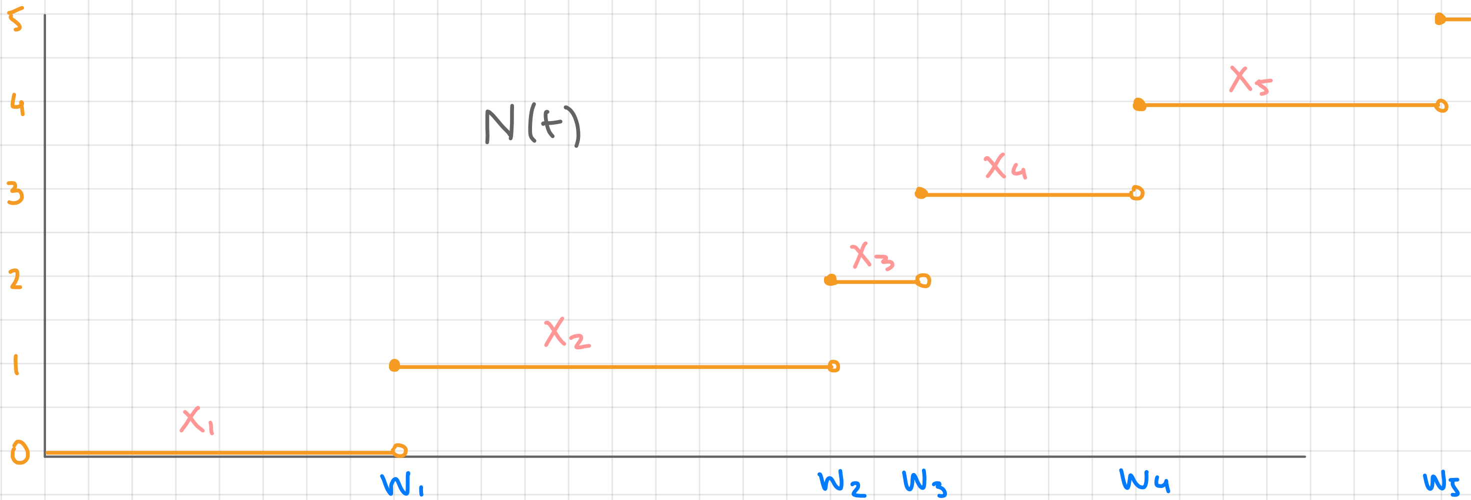
X_i are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \leq a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \leq b\}$$

Renewal process. Definition



$$\forall\, k \geq 0 \qquad N(t) \geq k \text{ if and only if } W_k \leq t \qquad (*)$$

Remarks 1) $(*)$ implies that $(N(t))_{t\geq 0}$ is determined by $(W_k)_{k\geq 0}$, so sometimes $(W_k)_{k\geq 0}$ is called renewal process

2) For any t, $\quad W_{N(t)} \leq t < W_{N(t)+1}$

Convolutions of c.d.f.s

Suppose that X and Y are independent r.v.s

$F: \mathbb{R} \to [0,1]$ is the c.d.f. of X (i.e. $P(X \le t) = F(t)$).

$G: \mathbb{R} \to [0,1]$ is the c.d.f. of Y

- if Y is discrete, then

$$F_{X+Y}(t) = P(X+Y \le t) = \sum_k P(X+Y \le t \mid Y=k)\, P(Y=k)$$

$$= \sum_k P(X+k \le t)\, P(Y=k) = \sum_k P(X \le t-k)\, P(Y=k)$$

$$= \sum_k F(t-k)\, P(Y=k) = \int_{-\infty}^{+\infty} F(t-x)\, dG(x) = F * G(t)$$

- if Y is continuous, then

$$F_{X+Y}(t) = P(X+Y \le t) = \int_{-\infty}^{+\infty} P(X+y \le t)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} F(t-y)\, f_Y(y)\, dy = \int_{-\infty}^{+\infty} F(t-y)\, dG(y) = F * G(t)$$

Distribution of W_k

Let $X_1, X_2, \dots$ be i.i.d. r.v.s, $X_i > 0$, and let $F : \mathbb{R} \to [0,1]$ be the c.d.f. of X_i (we call F the interoccurrence or interrenewal distribution). Then

- $F_1(t) := F_{W_1}(t) = P(W_1 \le t) = P(X_1 \le t) = F(t)$
- $F_2(t) := F_{W_2}(t) = F_{X_1+X_2}(t) = F * F(t)$
- $F_3(t) := F_{W_3}(t) = F_{(X_1+X_2)+X_3}(t) = (F * F) * F(t) =: F^{*3}(t)$
- More generally,

$$F_n(t) := F_{W_n}(t) = P(W_n \le t) = F^{*n}(t) \leftarrow \text{n-fold convolution of } F$$

Remark: $$F^{*(n+1)}(t) = \int_0^t F^{*n}(t-x)\, dF(x) = \int_0^t F(t-x)\, dF^{*n}(x)$$

Renewal function

Def. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. We call

$X \geq 0$, discrete

$$M(t) = E(N(t))$$

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

the renewal function.

Proposition 1. $M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{k=1}^{\infty} F^{*k}(t)$

Proof.

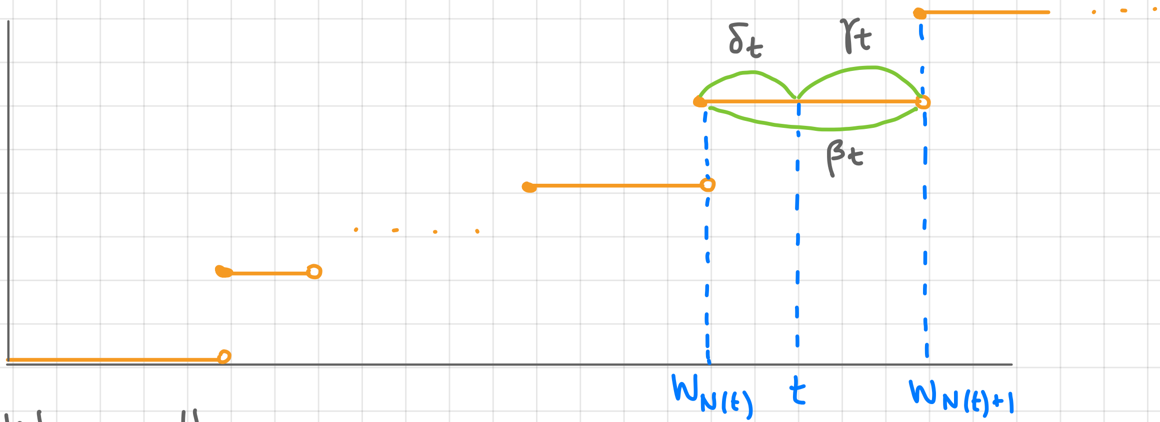
$$M(t) = E(N(t)) = \sum_{k=1}^{\infty} P(N(t) \geq k)$$

$$= \sum_{k=1}^{\infty} P(W_k \leq t)$$

$$= \sum_{k=1}^{\infty} F_k(t) = \sum_{k=1}^{\infty} F^{*k}(t) \quad \blacksquare$$

Related quantities

Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks

1) $\gamma_t > h$ iff $N(t+h) = N(t)$
2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\, E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) = E(X_1 + X_2 + \cdots + X_{N(t)+1})$

$E(X_2 + \cdots + X_{N(t)+1}) =$

$=$