MATH 180C HOMEWORK 3

SPRING 2022

Due date: **Saturday 4/23/2022 11:59 PM** (via Gradescope)
Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. *Pinsky and Karlin, Problem 6.3.3.*
   Let $(V_t)_{t\geq 0}$ be the two-state Markov chain whose transition probabilities are given by

$$P_{00}(t) = (1-\pi) + \pi e^{-\tau t}, \tag{1}$$
$$P_{01}(t) = \pi - \pi e^{-\tau t}, \tag{2}$$
$$P_{10}(t) = (1-\pi) - (1-\pi)e^{-\tau t}, \tag{3}$$
$$P_{11}(t) = \pi + (1-\pi)e^{-\tau t}. \tag{4}$$

   Suppose that the initial distribution is $(1-\pi, \pi)$. That is, assume that $P(V_0 = 0) = 1-\pi$, and $P(V_0 = 1) = \pi$.
   For $0 < s < t$, show that

$$E(V_s V_t) = \pi - \pi P_{10}(t-s), \tag{5}$$

   whence

$$\mathrm{Cov}(V_s, V_t) = \pi(1-\pi)e^{-\tau|t-s|}. \tag{6}$$

2. *Pinsky and Karlin, Exercise 6.4.6.*
   A birth and death process has parameters $\lambda_n = \lambda$ and $\mu_n = n\mu$, for $n = 0, 1, \ldots$. Determine the stationary distribution.

3. *Pinsky and Karlin, Problem 6.4.2.*
   Determine the stationary distribution, when it exists, for a birth and death process having constant parameters $\lambda_n = \lambda$ for $n = 0, 1, \ldots$ and $\mu_n = \mu$ for $n = 1, 2, \ldots$.

4. *Pinsky and Karlin, Problem 6.5.2.*
   Consider a birth and death process on the states $0, 1, \ldots, 5$ with parameters

$$\lambda_0 = \mu_0 = \lambda_5 = \mu_5 = 0, \tag{7}$$
$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \lambda_4 = 4, \tag{8}$$
$$\mu_1 = 4, \quad \mu_2 = 3, \quad \mu_3 = 2, \quad \mu_4 = 1. \tag{9}$$

   Note that 0 and 5 are absorbing states. Suppose the process begins in state $X_0 = 2$.
   (a) What is the probability of eventual absorption in state 0?

(b) What is the mean time to absorption?

5. *Pinsky and Karlin, Exercise 6.6.1.*

A certain type component has two states: 0=OFF and 1=OPERATING. In state 0, the process remains there a random length of time, which is exponentially distributed with parameter α, and then moves to state 1. The times in state 1 is exponentially distributed with parameter β, after which the process returns to 0.

The system has two of these components, A and B, with distinct parameters:

Component	**Operating Failure Rate**	**Repair Rate**
A	β_A	α_A
B	β_B	α_B

In order for the system to operate, at least one of components A and B must be operating (a parallel system). Assume that the component stochastic processes are independent of one another. Determine the long run probability that the system is operating by

(a) Considering each component separately as a two-state Markov chain and using their statistical independence;

(b) Considering the system as a four-state Markov chain and solving the equation $\pi Q = 0$.