MATH 142A: Introduction to Analysis

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Today: Limit theorems for sequences

> Q&A: January 20

Next: Ross § 10

Week 2:

- homework 2 (due Friday, January 22)
- Quiz 2 on Wednesday, January 20 (lectures 3-5)

Inequalities

- Cauchy-Schwarz-Bunyakovsky inequality

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

Proof: Denote $\sum_{k=1}^{n} a_k^2 =: A$, $\sum_{k=1}^{n} b_k^2 =: B$, $\sum_{k=1}^{n} a_k b_k =: C$.

① $A = 0$

② $A > 0$. Consider the function $p(x) :=$

Exercise $\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$

Inequalities

AM-GM inequality:

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

Proof ① If $a_1 a_2 \cdots a_n = 0$, then

② If $n = 1$, then

③ Suppose $n > 1$ and $\forall k$ $a_k > 0$. Then

Inequalities

- Bernoulli's inequality (L1):

$$\forall\, a \geq -1 \ \forall\, n \in \mathbb{N} \qquad (1+x)^n \geq 1+nx$$

- Triangle inequality (L2):

$$\forall\, a, b \in \mathbb{R} \qquad |a+b| \leq |a| + |b|$$

- Cauchy-Bunyakovsky-Schwarz inequality

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$$

- AM-GM inequality

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

$$G_n := \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} =: A_n$$

Limits and inequalities

Thm 9.11 (i) Let (a_n) and (b_n) be two convergent sequences, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

Then

(ii) Let (a_n), (b_n), (c_n) be three sequences such that $\exists N_0 \; \forall n > N_0 \quad a_n \le b_n \le c_n$.

Suppose that (a_n) and (c_n) are convergent, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} c_n = C$

Then

Proof (i). Choose Then

$\lim_{n\to\infty} a_n = A \Rightarrow$

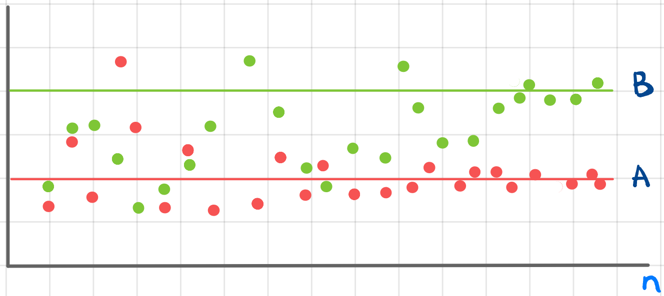
$\lim_{n\to\infty} b_n = B \Rightarrow$

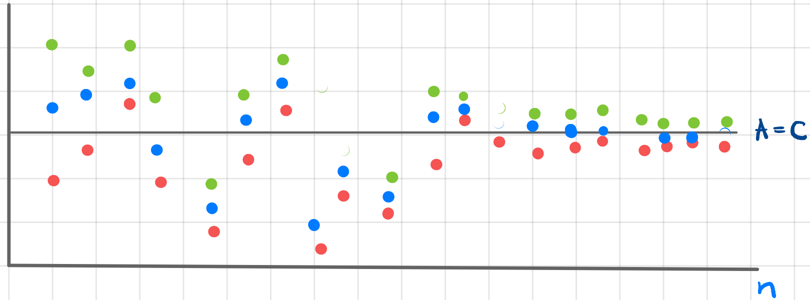
(ii)

$\lim_{n\to\infty} a_n = A \Rightarrow$

$\lim_{n\to\infty} c_n = C \Rightarrow$.

Limits and inequalities





Corollary 9.12 Suppose that $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

(i) $\exists N \ \forall n > N \ a_n > b_n$

(ii) $\exists N \ \forall n > N \ a_n \geq b_n$

(iii) $\exists N \ \forall n > N \ a_n > B$

(iv) $\exists N \ \forall n > N \ a_n \geq B$

Proof: Exercise (for (i) and (ii) use proof by contradiction).

Divergence to $\pm\infty$

Last time: $\lim_{n\to\infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = \lim_{n\to\infty} n \frac{5 - \frac{1}{n^4} - \frac{10}{n^5}}{7 - \frac{1}{n^2}} = ?$

Def 9.8. Let (s_n) be a sequence. We say that (s_n) diverges to $+\infty$ $(-\infty)$

We say that (s_n) has a limit, if it converges, or diverges to $+\infty$ or $-\infty$.

Example $\lim_{n\to\infty} \frac{5n^5 - n - 10}{7n^4 - n^2} = +\infty$

Proof.

Divergence to $\pm\infty$ and arithmetic operations

Thm 9.12 Let (s_n) be a sequence

(i) $\lim_{n\to\infty} s_n = +\infty$, $k>0$

(ii) $\lim_{n\to\infty} s_n = +\infty$

(iii) $\lim_{n\to\infty} s_n = +\infty$, $k<0$

Proof: Exercise

Thm 9.13 Let (s_n) and (t_n) be two sequences.

If $\lim_{n\to\infty} s_n = +\infty$ and $\inf\{t_n ; n\in\mathbb{N}\} > -\infty$, then

Proof. Fix $M>0$ and denote $m = \inf\{t_n : n\in\mathbb{N}\}$.

$\lim_{n\to\infty} s_n = +\infty \Rightarrow$

Examples

- $\lim_{n\to\infty} \left(n+\frac{1}{n}\right) = +\infty$
- $\lim_{n\to\infty} \left(n+\frac{1}{n}\right)^2 - n^2 = 2$
- $\lim_{n\to\infty} (n^2-n) = +\infty$
- $\lim_{n\to\infty} (n-n^2) = -\infty$

Divergence to $\pm\infty$ and arithmetic operations

Thm 9.9 Let (s_n) and (t_n) be sequences such that

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \left(\lim_{n\to\infty} t_n = t > 0 \quad \text{or} \quad \lim_{n\to\infty} t_n = +\infty \right)$$

Then

Proof (For $\lim_{n\to\infty} t_n = t > 0$) Fix $M > 0$. By Thm 9.11

Thm 9.10 Let (s_n) be a sequence such that $\forall n \; s_n > 0$. Then

Proof. $(\Rightarrow)$ Suppose $\lim_{n\to\infty} s_n = +\infty$.

$(\Leftarrow)$ Exercise.

Important examples

1. If $q \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{n^q} = 0$ (L4)

2. If $|a|<1$, then $\lim_{n\to\infty} a^n =$

Proof. ① If $a=0$, then $a^n = 0$, $\lim_{n\to\infty} 0 = 0$

② Let $a \neq 0$. Fix $\varepsilon > 0$.

Important examples

3. $\lim_{n \to \infty} \sqrt[n]{n} =$

<u>Proof</u>. ①

② Write

③

④

⑤

Important examples

4. $\forall\, a > 0 \quad \lim_{n\to\infty} \sqrt[n]{a} =$

Proof. If $a = 1$, then $\lim_{n\to\infty} 1 = 1$

If $a > 1$, then ①

②

③

If $a < 1$, denote $b = \frac{1}{a} > 1$. Then

①

②