# MATH 142A: Introduction to Analysis

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## Today: Limit theorems for sequences

> Q&A: January 20

## Next: Ross § 10

Week 3:

- homework 2 (due Friday, January 22)
- Quiz 2 on Wednesday, January 20 (lectures 3-5)

## Inequalities

- **Cauchy-Schwarz-Bunyakovsky inequality**

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2 \qquad (*)$$

Proof: Denote $\sum_{k=1}^{n} a_k^2 =: A$, $\sum_{k=1}^{n} b_k^2 =: B$, $\sum_{k=1}^{n} a_k b_k =: C$.

1. $A = 0 \Rightarrow \forall k \; a_k = 0 \Rightarrow C = 0 \qquad 0 \leq 0 \cdot B$, $(*)$ holds

2. $A > 0$. Consider the function $p(x) := Ax^2 + 2Cx + B$

$$p(x) = \sum_{k=1}^{n} a_k^2 x^2 + 2a_k b_k x + b_k^2 = \sum_{k=1}^{n} (a_k x + b_k)^2 \geq 0 \Rightarrow 4C^2 - 4AB \leq 0$$

$$\Rightarrow AB \geq C^2$$

$\blacksquare$

Exercise $\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$

$$\Leftrightarrow \exists \lambda, \mu \in \mathbb{R}, \; |\lambda| + |\mu| \neq 0 \quad \forall k \quad \lambda a_k = \mu b_k.$$

# Inequalities

## AM-GM inequality:

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

$$G_n := \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} =: A_n$$

Proof ① If $a_1 a_2 \cdots a_n = 0$, then $G_n = 0$, $A_n \ge 0$ ✓

② If $n = 1$, then $a_1 \le a_1$ ✓

③ Suppose $n > 1$ and $\forall k$ $a_k > 0$. Then $\forall k$ $A_k > 0$ and

$\forall k \in \{2, 3, \dots, n\}$ $\frac{A_k}{A_{k-1}} > 0$, which is equivalent to $\frac{A_k}{A_{k-1}} - 1 > -1$ $\forall k \in \{2, \dots, n\}$

Using Bernoulli's inequality: $$\left(\overset{1-1}{\frac{A_k}{A_{k-1}}}\right)^k \ge 1 + k\left(\frac{A_k}{A_{k-1}} - 1\right) = \frac{A_{k-1} + kA_k - kA_{k-1}}{A_{k-1}} = \frac{a_k}{A_{k-1}}$$

$\Rightarrow A_k^k \ge a_k A_{k-1}^{k-1}$. Apply for $k = n, n-1, n-2, \dots, 2$

$$A_n^n \ge a_n A_{n-1}^{n-1} \ge a_n a_{n-1} A_{n-2}^{n-2} \ge \cdots \ge a_n a_{n-1} \cdots a_2 a_1$$

■

# Inequalities

- **Bernoulli's inequality (L1):**

$$\forall\, a \geq -1 \ \forall\, n \in \mathbb{N} \qquad (1+x)^n \geq 1+nx$$

- **Triangle inequality (L2):**

$$\forall\, a, b \in \mathbb{R} \qquad |a+b| \leq |a| + |b|$$

- **Cauchy-Bunyakovsky-Schwarz inequality**

Let $n \in \mathbb{N}$ and $\{a_1, \dots, a_n, b_1, \dots, b_n\} \subset \mathbb{R}$. Then

$$\left( \sum_{k=1}^{n} a_k b_k \right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$$

- **AM-GM inequality**

Let $n \in \mathbb{N}$, $\{a_1, a_2, \dots, a_n\} \subset [0, +\infty)$. Then

$$G_n := \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} =: A_n$$

# Limits and inequalities

**Thm 9.11** (i) Let $(a_n)$ and $(b_n)$ be two convergent sequences, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

Then $A < B \Rightarrow \exists N \ \forall n > N \quad a_n < b_n$

(ii) Let $(a_n), (b_n), (c_n)$ be three sequences such that $\exists N_0 \ \forall n > N_0 \quad a_n \leq b_n \leq c_n$.

Suppose that $(a_n)$ and $(c_n)$ are convergent, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} c_n = C$

Then $A = C \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$

**Proof** (i). Choose $A < K < B$ Then

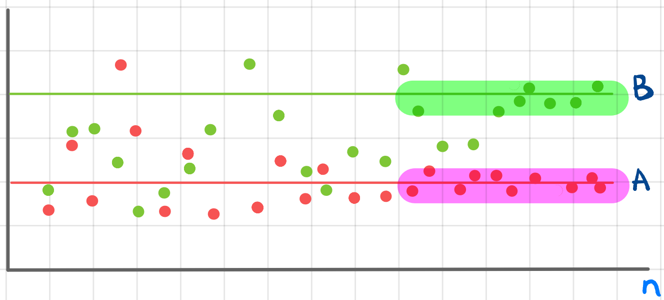
$$\left.\begin{array}{l} \lim_{n\to\infty} a_n = A \Rightarrow \exists N_1 \in \mathbb{N} \ \forall n > N_1 \ |a_n - A| < K - A > 0 \\ \lim_{n\to\infty} b_n = B \Rightarrow \exists N_2 \in \mathbb{N} \ \forall n > N_2 \ |b_n - B| < B - K > 0 \end{array}\right| \Rightarrow \begin{array}{l} \forall n > \max\{N_1, N_2\} \\ a_n < K - A + A = K - B + B < b_n \end{array}$$

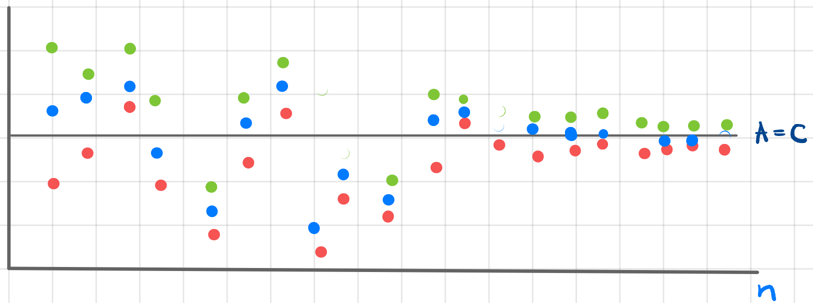
(ii) Fix $\varepsilon > 0$.

$$\left.\begin{array}{l} \lim_{n\to\infty} a_n = A \Rightarrow \exists N_1 \ \forall n > N_1 \ |a_n - A| < \varepsilon \\ \lim_{n\to\infty} c_n = C \Rightarrow \exists N_2 \ \forall n > N_2 \ |c_n - C| < \varepsilon \end{array}\right| \Rightarrow \left.\begin{array}{l} \forall n > N := \max\{N_0, N_1, N_2\} \\ A - \varepsilon < a_n \leq b_n \leq c_n < C + \varepsilon \end{array}\right| \Rightarrow \begin{array}{l} \forall n > N \\ |b_n - \underset{\overset{\|}{C}}{A}| < \varepsilon \end{array}$$

$\blacksquare$

# Limits and inequalities




Corollary 9.12 Suppose that $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

(i) $\exists N \ \forall n > N \ a_n > b_n \Rightarrow A \geq B$ $\qquad$ (e.g. $\forall n \ \frac{1}{n} > 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$)

(ii) $\exists N \ \forall n > N \ a_n \geq b_n \Rightarrow A \geq B$

(iii) $\exists N \ \forall n > N \ a_n > B \Rightarrow A \geq B$

(iv) $\exists N \ \forall n > N \ a_n \geq B \Rightarrow A \geq B$

Proof: Exercise (for (i) and (ii) use proof by contradiction).

# Divergence to $\pm\infty$

Last time: $\lim\limits_{n\to\infty} \dfrac{5n^5-n-10}{7n^4-n^2} = \lim\limits_{n\to\infty} n\, \dfrac{5-\frac{1}{n^4}-\frac{10}{n^5}}{7-\frac{1}{n^2}} = ?$

<u>Def</u> 9.8. Let $(s_n)$ be a sequence. We say that $(s_n)$ diverges to $+\infty$ $(-\infty)$

$$\lim_{n\to\infty} s_n = +\infty \quad \text{if } \forall M>0 \ \exists N\in\mathbb{N} \ \forall n>N \quad s_n > M$$

$$\lim_{n\to\infty} s_n = -\infty \quad \text{if } \forall M<0 \ \exists N\in\mathbb{N} \ \forall n>N \quad s_n < M$$

We say that $(s_n)$ has a limit, if it converges, or diverges to $+\infty$ or $-\infty$.

<u>Example</u> $\lim\limits_{n\to\infty} \dfrac{5n^5-n-10}{7n^4-n^2} = +\infty$

<u>Proof</u>. Fix $M>0$. If $n>1000$, then $\dfrac{5n^5-n-10}{7n^4-n^2} > \dfrac{4n^5}{8n^4} = \dfrac{n}{2}$ $[> M]$

Take $N = \max\{1000, [2M]\}$. Then $\forall n>N$

$$\frac{5n^5-n-10}{7n^4-n^2} > \frac{n}{2} > M$$

$\blacksquare$

# Divergence to $\pm\infty$ and arithmetic operations

**Thm 9.12** Let $(s_n)$ be a sequence

(i) $\lim_{n\to\infty} s_n = +\infty$, $k>0$ $\Rightarrow$ $\lim_{n\to\infty}(k\cdot s_n) = +\infty$

(ii) $\lim_{n\to\infty} s_n = +\infty$ $\Leftrightarrow$ $\lim_{n\to\infty}(-s_n) = -\infty$

(iii) $\lim_{n\to\infty} s_n = +\infty$, $k<0$ $\Rightarrow$ $\lim_{n\to\infty}(k\cdot s_n) = -\infty$

Proof: Exercise

**Thm 9.13** Let $(s_n)$ and $(t_n)$ be two sequences.

If $\lim_{n\to\infty} s_n = +\infty$ and $\inf\{t_n ; n\in\mathbb{N}\} > -\infty$, then $\lim_{n\to\infty}(s_n+t_n) = +\infty$

**Proof** Fix $M>0$ and denote $m = \inf\{t_n : n\in\mathbb{N}\}$.

$\lim_{n\to\infty} s_n = +\infty \Rightarrow \exists N\ \forall n>N\ \ s_n > M-m$. Then $\forall n>N$ $s_n+t_n > s_n+m > M-m+m$
$= M$

**Examples**

- $\lim_{n\to\infty}\left(n+\frac{1}{n}\right) = +\infty$
- $\lim_{n\to\infty} \left(n+\frac{1}{n}\right)^2 - n^2 = 2$
- $\lim_{n\to\infty}(n^2-n) = +\infty$
- $\lim_{n\to\infty}(n-n^2) = -\infty$

# Divergence to $\pm\infty$ and arithmetic operations

**Thm 9.9** Let $(s_n)$ and $(t_n)$ be sequences such that

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \left( \lim_{n\to\infty} t_n = t > 0 \quad \text{or} \quad \lim_{n\to\infty} t_n = +\infty \right)$$

Then $$\lim_{n\to\infty} (s_n t_n) = +\infty$$

**Proof** (For $\lim_{n\to\infty} t_n = t > 0$) Fix $M > 0$. By Thm 9.11 $\exists N_1 \; \forall n > N_1 \; t_n > \frac{t}{2} > 0$

$\lim_{n\to\infty} s_n = +\infty \Rightarrow \exists N_2 \; \forall n > N_2 \; s_n > M \cdot \frac{2}{t}$. Then

$$\forall n > \max\{N_1, N_2\} \quad s_n t_n > s_n \cdot \frac{t}{2} > M \cdot \frac{2}{t} \cdot \frac{t}{2} = M \qquad \blacksquare$$

**Thm 9.10** Let $(s_n)$ be a sequence such that $\forall n \; s_n > 0$. Then

$$\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \left(\frac{1}{s_n}\right) = 0$$

**Proof**. $(\Rightarrow)$ Suppose $\lim_{n\to\infty} s_n = +\infty$. Fix $\varepsilon > 0$. $\left[\exists N \; \forall n > N \; \frac{1}{s_n} < \varepsilon \left(\iff s_n > \frac{1}{\varepsilon}\right)\right]$

$\lim_{n\to\infty} s_n = +\infty \Rightarrow \exists N \; \forall n > N \; s_n > \frac{1}{\varepsilon}$. Then

$$\forall n > N \quad \left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} < \varepsilon \qquad \blacksquare$$

$(\Leftarrow)$ Exercise.

# Important examples

1. If $q \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{n^q} = 0$ (L4)

2. If $|a|<1$, then $\lim_{n\to\infty} a^n = 0$

Proof. ① If $a=0$, then $a^n = 0$, $\lim_{n\to\infty} 0 = 0$

② Let $a \neq 0$. Fix $\varepsilon > 0$. [ $\exists N \ \forall n \geq N \ |a^n - 0| = |a^n| = |a|^n < \varepsilon$ ]

Denote $b = \frac{1}{|a|} - 1 > 0$ ( so that $|a| = \frac{1}{1+b}$ )

By Bernoulli's inequality

$$\left(\frac{1}{|a|}\right)^n = (1+b)^n \geq 1 + nb > nb$$

and thus $|a^n - 0| = |a^n| < \frac{1}{nb}$ [enough $\frac{1}{nb} < \varepsilon \iff n > \frac{1}{\varepsilon b}$ ]

Take $N = \left[\frac{1}{\varepsilon b}\right]$. Then $\forall n \geq N \ |a^n - 0| < \frac{1}{Nb} < \varepsilon$ ∎

## Important examples

3. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Proof. ① $\forall n \quad \sqrt[n]{n} \geq 1$

② Write $n = \sqrt{n}\cdot\sqrt{n}\cdot\overbrace{1\cdot 1\cdot \dots \cdot 1}^{n-2}$. Then by AM-GM inequality

$$\sqrt[n]{n} = \sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdots 1}_{n-2}} \leq \frac{\sqrt{n}+\sqrt{n}+\overbrace{1+\dots+1}^{n-2}}{n} = \frac{2\sqrt{n}+n-2}{n} = 1+\frac{2}{\sqrt{n}}-\frac{2}{n}$$

③ $\lim\limits_{n\to\infty}\sqrt{n} = +\infty$ [By AP $\forall M>0\ \exists N\ \forall n>N\ \ n>M^2 \Leftrightarrow \sqrt{n}>M$]

④ By Thm 9.10, $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0 \Rightarrow \lim\limits_{n\to\infty}\left(1+\frac{2}{\sqrt{n}}-\frac{2}{n}\right) = 1$

⑤ $1 \leq \sqrt[n]{n} \leq 1+\frac{2}{\sqrt{n}}-\frac{2}{n}$, by Thm 9.11 and ④ $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$. $\blacksquare$

## Important examples

4. $\forall a > 0 \quad \lim_{n\to\infty} \sqrt[n]{a} = 1$

Proof. If $a=1$, then $\lim_{n\to\infty} 1 = 1$

If $a>1$, then ① $\forall n \quad \sqrt[n]{a} \geq 1$

② $\forall n > a \quad \sqrt[n]{a} \leq \sqrt[n]{n}$

③ By Thm 9.11 $\quad \lim_{n\to\infty} \sqrt[n]{a} = 1$

If $a<1$, denote $b = \frac{1}{a} > 1$. Then

① $\sqrt[n]{a} = \frac{1}{\sqrt[n]{b}}$

② $\lim_{n\to\infty} \sqrt[n]{b} = 1$

Thm 9.5 $\Rightarrow$

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{b}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{b}} = \frac{1}{1} = 1.$$

$\blacksquare$