MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Ordered field

> Q&A: January 8

Next: Ross § 4

Week 1:

- visit course website
- homework 0 (due Friday, January 8)
- join Piazza

Fields

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ (proper subsets)

Let $\mathbb{F}$ be a set with two binary operations

$: \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $: \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

Consider the following properties:

A1. $\forall\, a, b, c \in \mathbb{F}$

A2. $\forall\, a, b \in \mathbb{F}$

A3. $\mathbb{F}$ s.t. $\forall\, a \in \mathbb{F}$

A4. $\forall\, a \in \mathbb{F}$ $\in \mathbb{F}$ s.t.

$\mathbb{Q}_{\geq 0} := \{ r \in \mathbb{Q} : r \geq 0 \}$

Fields (cont)

M1. $\forall a, b, c \in \mathbb{F}$ (associativity)

M2. $\forall a, b \in \mathbb{F}$ (commutativity)

M3. $\exists$ $\mathbb{F}$ s.t. $\forall a \in \mathbb{F}$ (neutral element)

M4. $\forall a \in \mathbb{F}$ s.t. $a \neq 0$ $\in \mathbb{F}$ s.t. (multiplicative inverse)

DL $\forall a, b, c \in \mathbb{F}$

Definition (Field) Set $\mathbb{F}$ with binary operations $+$ and $\cdot$ satisfying A1–A4, M1–M4, DL is called a

A1–A4, M1-M4 and DL are called the

Remark $\mathbb{Q}$, $\mathbb{R}$ are fields, $\mathbb{N}$, $\mathbb{Z}$ are not fields (with usual $+, \cdot$)

Consequences of field axioms

Theorem 3.1 Let $\mathbb{F}$ with operations $+$ and $\cdot$ be a field. Then for any $a, b, c \in \mathbb{F}$

(i) $a + c = b + c \Rightarrow a = b$

(ii) $a \cdot 0 = 0$

(iii) $(-a)b = -ab$

(iv) $(-a)(-b) = ab$

(v) $ac = bc \wedge c \neq 0 \Rightarrow a = b$

(vi) $ab = 0 \Rightarrow a = 0 \vee b = 0$

Proof. (i)

which implies that

(ii)

Prop

Proof.

Ordered fields

Definition Set S with a (binary) relation $\leq$ is called

if

(O1) $\forall\ a, b \in S$

(O2) $\forall\ a, b \in S$

(O3) $\forall\ a, b, c \in S$

Definition Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ and order relation $\leq$. $\mathbb{F}$ is called an if

- $\mathbb{F}$ with $+$ and $\cdot$ is a
- $\mathbb{F}$ with $\leq$ is
- (O4) $\forall\ a, b, c \in \mathbb{F}$
- (O5)

Properties of ordered fields

Theorem 3.2 Let $\mathbb{F}$ be an ordered field with operations $+, \cdot$ and order relation $\leq$. Then $\forall\, a, b, c$ in $\mathbb{F}$

(i) $a \leq b \Rightarrow -b \leq -a$

(ii) $a \leq b \wedge c \leq 0 \Rightarrow bc \leq ac$

(iii) $0 \leq a \wedge 0 \leq b \Rightarrow 0 \leq ab$

(iv) $0 \leq a^2$ $[a^2 = a \cdot a]$

(v) $0 < 1$

(vi) $0 < a \Rightarrow 0 < a^{-1}$

(vii) $0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}$

["$a < b$" means "$a \leq b \wedge a \neq b$"]

Proof. (i)

(ii)

(iv)

Absolute value

Let $\mathbb{F}$ be an ordered field

Def 3.3. Let $a \in \mathbb{F}$. We call $|a| := \{$

the **absolute value** of a.

Def 3.4 Let $a, b \in \mathbb{F}$. We call $\mathrm{dist}(a,b) := |a-b|$

the **distance** between a and b $[a-b := a+(-b)]$

Thm 3.5 (i) $\forall a \in \mathbb{F}$

(ii) $\forall a, b \in \mathbb{F}$

(iii) $\forall a, b \in \mathbb{F}$ (Triangle inequality)

Proof (i) Follows from the definition and Thm 3.2 (i).

(ii) Exercise (check 4 cases)

Proof (cont) (iii)

Step 1: $\forall c \in \mathbb{F}$, $0 \leq c \Rightarrow -|c| \leq c \leq |c|$

Proof:

Step 2: $\forall c \in \mathbb{F}$, $c \leq 0 \Rightarrow -|c| \leq c \leq |c|$

Proof: $c \leq 0 \Rightarrow (|c| = -c) \wedge (-|c| = c) \wedge (0 \leq |c|) \Rightarrow -|c| \leq c \leq 0 \leq |c|$

Step 3: $-|a| \leq a \leq |a|$, $-|b| \leq b \leq |b|$

Follows from Step 1 and Step 2.

Step 4: $-|a| - |b| \leq a - |b| \leq a + b \leq$

Corollary $\forall a, b, c \in \mathbb{F}$

Proof. Exercise