MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Ordered field

> Q&A: January 8

Next: Ross § 4

Week 1:

- visit course website
- homework 0 (due Friday, January 8)
- join Piazza

Fields

$$\mathbb{N} \subset \mathbb{Z} \subset \underline{\underline{\mathbb{Q}}} \subset \underline{\underline{\mathbb{R}}} \quad \text{(proper subsets)}$$

Let $\mathbb{F}$ be a set with two binary operations

$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (addition) and $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (multiplication)

Consider the following properties:

A1. $a + (b + c) = (a + b) + c \quad \forall\, a, b, c \in \mathbb{F}$ (associativity)

$(1 : 2) : 2 \neq 1 : (2 : 2)$, $\quad 2^{2^3} = 2^{(2^3)} \neq (2^2)^3$ ← not associative

A2. $a + b = b + a \quad \forall\, a, b \in \mathbb{F}$ (commutativity) ["$\forall$" means "for all"]

$3 - 2 \neq 2 - 3$ ← not commutative

A3. $\exists\, 0 \in \mathbb{F}$ s.t. $a + 0 = a \quad \forall\, a \in \mathbb{F}$ (neutral element)

$0 \notin \mathbb{N}$ ["$\exists$" means "there exists"]

A4. $\forall\, a \in \mathbb{F} \;\; \exists\, (-a) \in \mathbb{F}$ s.t. $a + (-a) = 0$ (additive inverse of a)

$\mathbb{Q}_{\geq 0} := \{ r \in \mathbb{Q} : r \geq 0 \}$ $\qquad -1 \notin \mathbb{Q}_{\geq 0}$

Fields (cont)

corrected

M1. $a(bc) = (ab)c \quad \forall\, a,b,c \in \mathbb{F}$ (associativity)

M2. $ab = ba \quad \forall\, a,b \in \mathbb{F}$ (commutativity)

M3. $\exists\, 1 \in \mathbb{F}$ s.t. $a \cdot 1 = a \quad \forall\, a \in \mathbb{F}$ (neutral element)

M4. $\forall\, a \in \mathbb{F}$ s.t. $a \neq 0$ $\exists\, a^{-1} \in \mathbb{F}$ s.t. $a a^{-1} = 1$ (multiplicative inverse)

$\mathbb{F} = \{ M \in \mathbb{R}^{2\times 2} : \det M \neq 0 \}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

DL $a(b+c) = ab + ac \quad \forall\, a,b,c \in \mathbb{F}$

<u>Definition (Field)</u> Set $\mathbb{F}$ with <u>more than one element</u> and binary operations $+$ and $\cdot$ satisfying A1–A4, M1–M4, DL is called a **field**.

A1–A4, M1–M4 and DL are called the **field axioms**

<u>Remark</u> $\mathbb{Q}$, $\mathbb{R}$ are fields, $\mathbb{N}$, $\mathbb{Z}$ are not fields (with usual $+, \cdot$)

Consequences of field axioms

Theorem 3.1 Let $\mathbb{F}$ with operations $+$ and $\cdot$ be a field.

Then for any $a, b, c \in \mathbb{F}$

(i) $a+c = b+c \Rightarrow a = b$

(ii) $a \cdot 0 = 0$

(iii) $(-a)b = -ab$

(iv) $(-a)(-b) = ab$

(v) $ac = bc \wedge c \neq 0 \Rightarrow a = b$

(vi) $ab = 0 \Rightarrow a = 0 \vee b = 0$

Proof. (i) $a+c = b+c \overset{A4}{\Rightarrow} (a+c)+(-c) = (b+c)+(-c)$

$(a+c)+(-c) \overset{A1}{=} a+(c+(-c)) \overset{A4}{=} a+0 \overset{A3}{=} a$, $(b+c)+(-c) \overset{A1}{=} b+(c+(-c)) \overset{A4}{=} b+0 \overset{A3}{=} b$

which implies that $a = b$

(ii) $a \cdot 0 \overset{A3}{=} a \cdot (0+0) \overset{DL}{=} a \cdot 0 + a \cdot 0$

$a \cdot 0 \overset{A3}{=} a \cdot 0 + 0 \overset{A2}{=} 0 + a \cdot 0$

$\Rightarrow a \cdot 0 + a \cdot 0 = 0 + a \cdot 0 \overset{(i)}{\Rightarrow} a \cdot 0 = 0$ ∎

Prop If 0_1 and 0_2 are (additive) neutral elements, then $0_1 = 0_2$.

Proof. $0_1 \overset{A3}{=} 0_1 + 0_2 \overset{A2}{=} 0_2 + 0_1 \overset{A3}{=} 0_2$ ∎

Ordered fields

<u>Definition</u> Set S with a (binary) relation $\leq$ is called **linearly ordered** if

(O1) $\forall\, a, b \in S$ either $a \leq b$ or $b \leq a$

(O2) $\forall\, a, b \in S$ $(a \leq b \wedge b \leq a \Rightarrow a = b)$ [antisymmetry]

(O3) $\forall\, a, b, c \in S$ $(a \leq b \wedge b \leq c \Rightarrow a \leq c)$ [transitivity]

<u>Definition</u> Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ and order relation $\leq$. $\mathbb{F}$ is called an **ordered field** if

- $\mathbb{F}$ with $+$ and $\cdot$ is a **field**
- $\mathbb{F}$ with $\leq$ is **linearly ordered**
- (O4) $a \leq b \Rightarrow a + c \leq b + c \quad \forall\, a, b, c \in \mathbb{F}$
- (O5) $a \leq b \wedge 0 \leq c \Rightarrow ac \leq bc$

Properties of ordered fields

Theorem 3.2 Let $\mathbb{F}$ be an ordered field with operations $+, \cdot$ and order relation $\leq$. Then $\forall\, a, b, c$ in $\mathbb{F}$

(i) $a \leq b \Rightarrow -b \leq -a$

(ii) $a \leq b \wedge c \leq 0 \Rightarrow bc \leq ac$

(iii) $0 \leq a \wedge 0 \leq b \Rightarrow 0 \leq ab$

(iv) $0 \leq a^2$ $[a^2 = a \cdot a]$

(v) $0 < 1$

(vi) $0 < a \Rightarrow 0 < a^{-1}$

(vii) $0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}$

["$a < b$" means "$a \leq b \wedge a \neq b$"]

Proof. (i) $a \leq b \overset{O4}{\Rightarrow} a + ((-a)+(-b)) \leq b + ((-a)+(-b)) \overset{A1-A4}{\Rightarrow} -b \leq -a$

(ii) $0+0=0 \Rightarrow -0=0$, therefore $c \leq 0 \overset{(i)}{\Rightarrow} 0 \leq -c$. Then

$a \leq b \wedge 0 \leq -c \overset{O5}{\Rightarrow} a(-c) \leq b(-c) \overset{T3.1}{\Rightarrow} -ac \leq -bc \overset{(i)}{\Rightarrow} bc \leq ac$

(iv) By O1 either $a \leq 0$ or $0 \leq a$. $0 \leq a \overset{O5}{\Rightarrow} 0 \cdot a \leq a \cdot a \Rightarrow 0 \leq a^2$

$a \leq 0 \Rightarrow 0 \leq (-a)(-a) \overset{T3.1}{\Rightarrow} 0 \leq a^2$

$\blacksquare$

Absolute value

Let $\mathbb{F}$ be an ordered field

Def 3.3. Let $a \in \mathbb{F}$. We call $|a| := \begin{cases} a & \text{if } 0 \le a \\ -a & \text{if } a \le 0 \end{cases}$

the absolute value of a.

Def 3.4 Let $a, b \in \mathbb{F}$. We call $\operatorname{dist}(a,b) := |a-b|$

the distance between a and b $\quad [a-b := a+(-b)]$

Thm 3.5 (i) $0 \le |a| \quad \forall a \in \mathbb{F}$

(ii) $|ab| = |a|\,|b| \quad \forall a, b \in \mathbb{F}$

(iii) $|a+b| \le |a| + |b| \quad \forall a, b \in \mathbb{F}$ (Triangle inequality)

Proof (i) Follows from the definition and Thm 3.2 (i).

(ii) Exercise (check 4 cases)

Proof (cont) (iii)

Step 1: $\forall c \in \mathbb{F}$, $0 \leq c \Rightarrow -|c| \leq c \leq |c|$

Proof: $0 \leq c \Rightarrow |c| = c \wedge -c \leq 0 \overset{O1}{\Rightarrow} -|c| \leq 0 \leq c \leq |c|$

Step 2: $\forall c \in \mathbb{F}$, $c \leq 0 \Rightarrow -|c| \leq c \leq |c|$

Proof: $c \leq 0 \Rightarrow (|c| = -c) \wedge (-|c| = c) \wedge (0 \leq |c|) \Rightarrow -|c| \leq c \leq 0 \leq |c|$

Step 3: $-|a| \leq a \leq |a|$, $-|b| \leq b \leq |b|$

Follows from Step 1 and Step 2.

Step 4: $-|a| - |b| \leq a - |b| \leq a + b \leq |a| + b \leq |a| + |b|$

$$\Rightarrow \left| \begin{array}{l} a+b \leq |a|+|b| \\ -(a+b) \leq -(-|a|-|b|) = |a|+|b| \end{array} \right| \Rightarrow |a+b| \leq |a|+|b|$$

$\blacksquare$

Corollary $\forall a, b, c \in \mathbb{F}$ $\quad dist(a,c) \leq dist(a,b) + dist(b,c)$

Proof. Exercise (Hint: Define $x = a-b$, $y = b-c$)