MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Mean Value Theorem

> Q&A: March 1

Next: Ross § 30

Week 8:

- Homework 8 (due Sunday, March 7)

Fermat's Theorem

Thm 29.1 (i) $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$
(ii) f assumes its max or min at x_0
(iii) $f'(x_0)$ exists
$\Rightarrow$

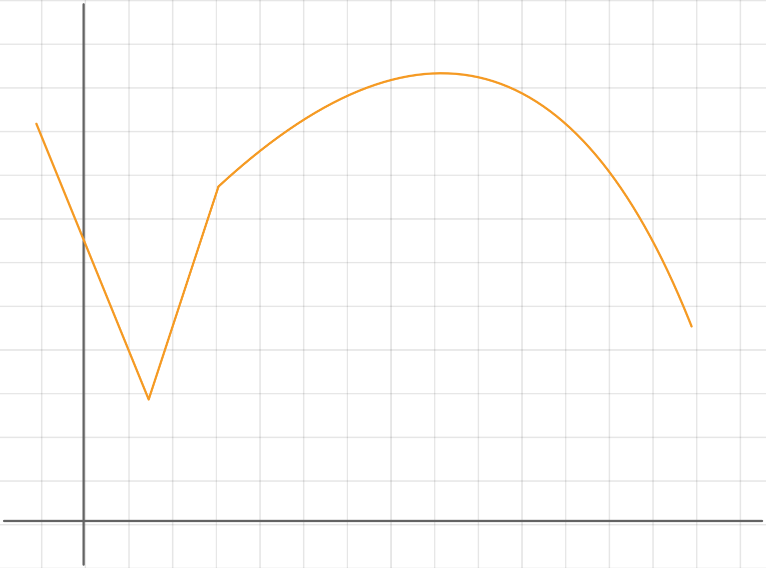
Proof. Suppose that f assumes its max at x_0 (otherwise take $-f$)

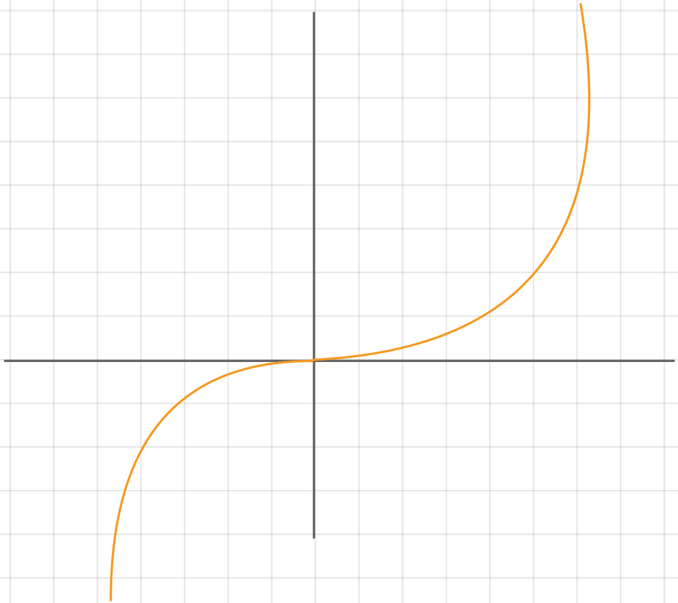
If , then

so $\forall x\in(x_0,x_0+\delta)$

Therefore, . Similar argument shows that

Critical points





Rolle's Theorem

Notation: If $S \subset \mathbb{R}$ then

- $f \in C(S)$ means that f is continuous on S
- $f \in D(S)$ means that f is differentiable on S

Thm 29.2

$$\left.\begin{array}{l} \text{(i) } f \in C([a,b]) \\ \text{(ii) } f \in D((a,b)) \\ \text{(iii) } f(a) = f(b) \end{array}\right| \Rightarrow$$

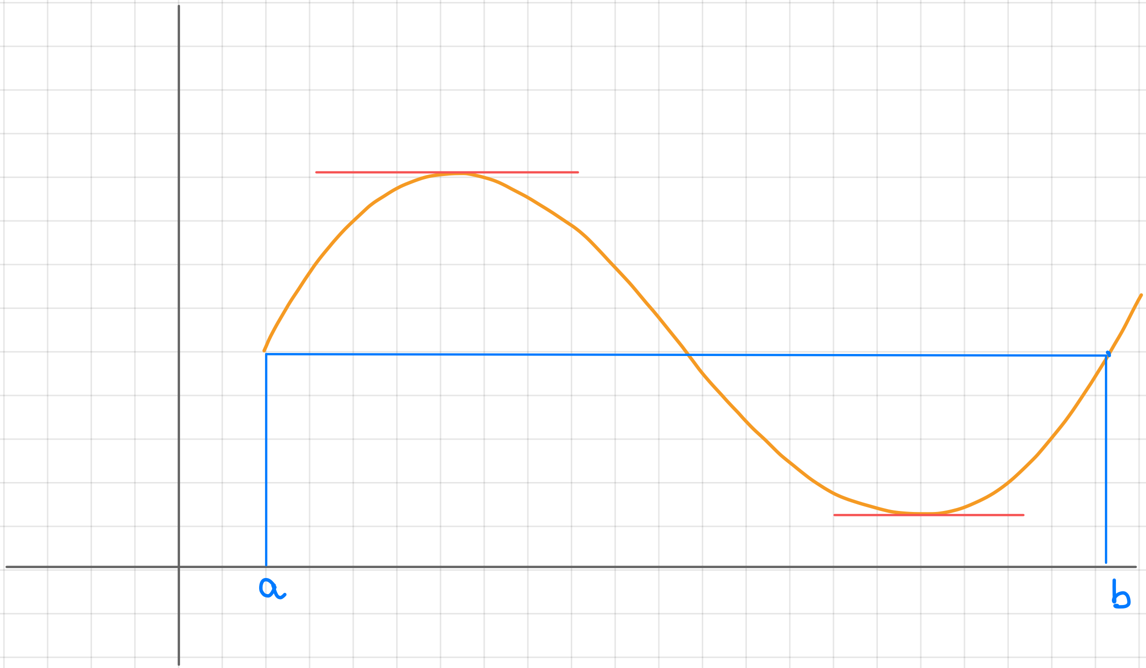
Proof. By the maximum-value theorem (Thm 18.1)

If $\{x_0, y_0\} = \{a, b\}$, then

If $y_0 \in (a,b)$, then by Thm 29.1

If $x_0 \in (a,b)$, then by Thm 29.1

Rolle's Theorem



Mean-value Theorem (Lagrange's Theorem)

Thm 29.3

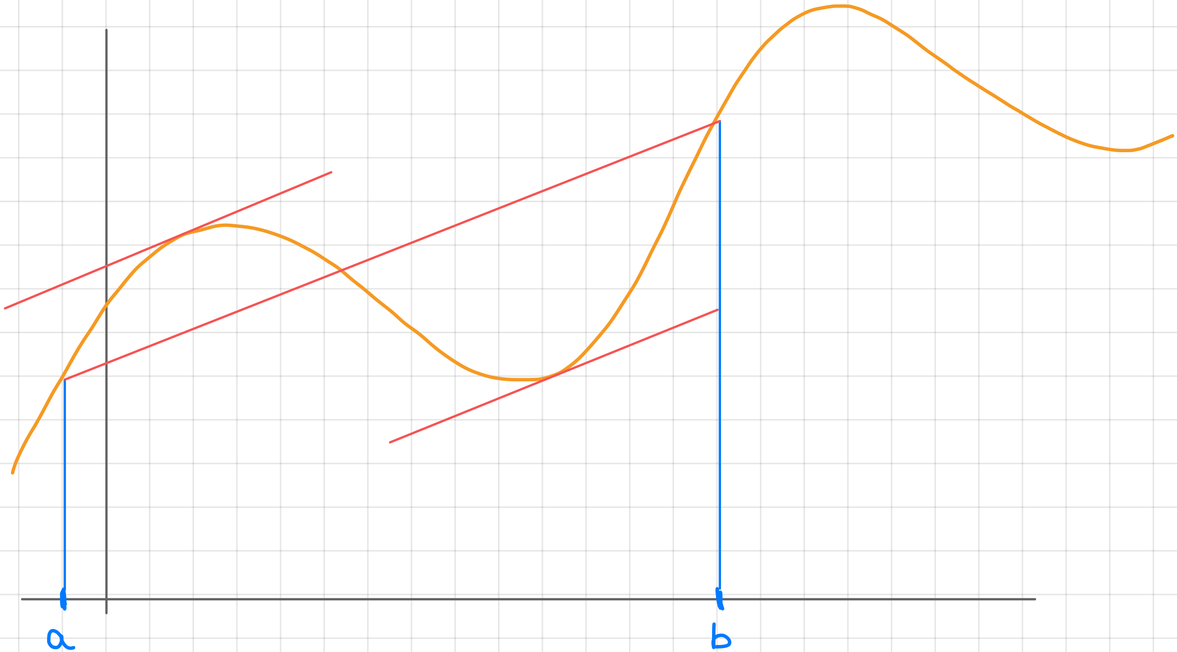
$$\left.\begin{array}{l} \text{(i) } f \in C([a,b]) \\ \text{(ii) } f \in D((a,b)) \end{array}\right| \Rightarrow$$

Proof. Denote $F : [a,b] \to \mathbb{R}$, $F(x) =$

Then

Since $F'(c) =$, we get

Mean-value Theorem (Lagrange's Theorem)



Corollaries

Cor. 29.4 (i) $f \in D((a,b))$
(ii) $f' = 0$ on (a,b) $\Bigg| \Rightarrow$

Proof (By contradiction). If $\exists\, x, y \in (a,b)$ s.t. $f(x) \neq f(y)$,
then by Lagrange's Thm

Cor 29.5 (i) $f, g \in D((a,b))$
(ii) $f' = g'$ on (a,b) $\Bigg| \Rightarrow$

Proof Apply Cor. 29.4 to $f - g$:

Application of Thms 29.1-29.3

1) $\forall\ x, y \in \mathbb{R}$

Fix $x, y \in \mathbb{R}$, $x < y$. $\sin \in C([x,y])$, $\sin \in D((x,y))$, so by Lagrange's thm

and thus

2) $\forall\ x, y \in [1, +\infty)$

Fix $x, y \in [1, +\infty)$, $x < y$. Let $f : [0, +\infty) \to [0, +\infty)$, $f(u) = \sqrt{u}$. Then

$f \in C([x,y])$, $f \in D((x,y))$, so by Lagrange's Thm

, and thus

Application of Thms 29.1-29.3

3) $\forall x \in \mathbb{R}$

Let $x>0$, $f(u)=e^u$. $f \in C([0,x])$, $f \in D((0,x))$, $f'(u)=e^u$, so by Lagrange's thm

If $x<0$, apply Lagrange's thm to $f \in C([x,0])$, $f \in D((x,0))$.

Then

Therefore,

Monotonic functions and the mean-value theorem

Def. 29.6 Let $I \subset \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$. We say that

- f is strictly increasing on I if $\forall\, x, y \in I \ (x < y \Rightarrow f(x) < f(y))$
- f is strictly decreasing on I if $\forall\, x, y \in I \ (x < y \Rightarrow f(x) > f(y))$
- f is increasing on I if $\forall\, x, y \in I \ (x < y \Rightarrow f(x) \leq f(y))$
- f is decreasing on I if $\forall\, x, y \in I \ (x < y \Rightarrow f(x) \geq f(y))$

Cor 29.7. $f \in D((a,b))$. Then

(i) f is strictly increasing on (a,b) if for all $x \in (a,b)$

(ii) f is strictly decreasing on (a,b) if for all $x \in (a,b)$

(iii) f is increasing on (a,b) if for all $x \in (a,b)$

(iv) f is decreasing on (a,b) if for all $x \in (a,b)$

Proof. (ii) Take $x, y \in (a,b)$, $x < y$. By Lagrange's thm

Intermediate-value theorem for derivatives (Darboux's Thm)

Thm 29.8 $f \in D((a,b))$, $x_1, x_2 \in (a,b)$, $x_1 < x_2$.

(i) $f'(x_1) < f'(x_2) \Rightarrow \forall c \in (f'(x_1), f'(x_2))\ \exists\ x \in (x_1, x_2)$ s.t. $f'(x) = c$

(ii) $f'(x_1) > f'(x_2) \Rightarrow \forall c \in (f'(x_2), f'(x_1))\ \exists\ x \in (x_1, x_2)$ s.t. $f'(x) = c$

Proof : (i) Fix $c \in (f'(x_1), f'(x_2))$.

Consider $g(x) =$ Then

① , by Thm 18.1 (max-value)

②

$$\lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} < 0$$

Similarly,

③ $\xRightarrow{\text{Fermat's Thm}}$