

MATH 142A: Introduction to Analysis

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Today: Limits of functions

> Q&A: February 22

Next: Ross § 28

Week 8:

- Homework 7 (due Sunday, February 28)
- Midterm 2 (Wednesday, February 24): Lectures 8-16

Limit of a function, ε - δ definition

D 20.12 Let f be a function defined on $S \subset \mathbb{R}$, let $a \in \mathbb{R}$ be a limit of some sequence in S , let $L \in \mathbb{R}$. We say that f tends to L as x tends to a along S if

Thm 20.6 Definitions 20.1 and 20.12 are equivalent.

Proof (\Rightarrow) Suppose that (*) does not hold:

(\Leftarrow) Let (x_n) be a sequence, $\forall n \ x_n \in S$, $\lim x_n = a$.

Fix $\varepsilon > 0$. Take δ as in (*). $\lim x_n = a \Rightarrow$

By (*) $\forall n > N$

Limit of a function, ε - δ definition

D 20.13 Suppose that f is defined on $(a-c, a+c) \setminus \{a\}$ for some $c > 0$.

(a) We say that L is the (two-sided) limit of f at a if $a, L \in \mathbb{R}$

(b) We say that L is the right-hand limit of f at a if

(c) We say that L is the left-hand limit of f at a if

Corollary 20.7-20.8 Definitions 20.3 (a), (b), (c) and 20.13 (a), (b), (c) are equivalent.

Proof Follows from Thm 20.6 by specializing

(a) , (b) , (c)

Limit of a function

Suppose $f: S \rightarrow \mathbb{R}$, $a, L \in \mathbb{R}$

• $\lim_{x \rightarrow +\infty} f(x) = L \stackrel{\text{Def}}{\Leftrightarrow}$

• $\lim_{x \rightarrow +\infty} f(x) = +\infty \stackrel{\text{Def}}{\Leftrightarrow}$

• $\lim_{x \rightarrow +\infty} f(x) = -\infty \stackrel{\text{Def}}{\Leftrightarrow}$

• $\lim_{x \rightarrow -\infty} f(x) = L \stackrel{\text{Def}}{\Leftrightarrow} \forall \varepsilon > 0 \exists t > 0 (x < -t \Rightarrow |f(x) - L| < \varepsilon)$

• $\lim_{x \rightarrow -\infty} f(x) = +\infty \stackrel{\text{Def}}{\Leftrightarrow} \forall M > 0 \exists t > 0 (x < -t \Rightarrow f(x) > M)$

• $\lim_{x \rightarrow a} f(x) = +\infty \stackrel{\text{Def}}{\Leftrightarrow} \forall M > 0 \exists \delta > 0 (0 < |x - a| < \delta \Rightarrow f(x) > M)$

• $\lim_{x \rightarrow a^-} f(x) = +\infty \stackrel{\text{Def}}{\Leftrightarrow} \forall M > 0 \exists \delta > 0 (x \in (a - \delta, a) \Rightarrow f(x) > M)$

Two-sided limits and left-hand/right-hand limits

Thm 20.10

Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing $a \in \mathbb{R}$. Let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = L \wedge \lim_{x \rightarrow a^-} f(x) = L$$

Proof. (\Rightarrow) Exercise

(\Leftarrow) Suppose $L \in \mathbb{R}$. Fix $\varepsilon > 0$.

Suppose $L = +\infty$. Fix $M > 0$.

Examples

1)

2) Let $a > 1$, $p \in \mathbb{N}$, $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) =$. Then

Fix $\epsilon > 0$. By IEG

Then

and

Squeeze Lemma

Thm. 20.14 Let $f, g, h: S \rightarrow \mathbb{R}$, $\forall x \in S$ $f(x) \leq g(x) \leq h(x)$

Let $a, L \in \mathbb{R} \cup \{+\infty, -\infty\}$.

If $\lim_{S \ni x \rightarrow a} f(x) = \lim_{S \ni x \rightarrow a} h(x) = L$, then

Proof. Take any sequence (s_n) in S s.t. $\lim s_n = a$. Then

IE 12 $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$. Fix $\varepsilon > 0$. By IE from Lecture 7,

and thus

\Rightarrow