MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Limits of functions

> Q&A: February 19

Next: Ross § 20

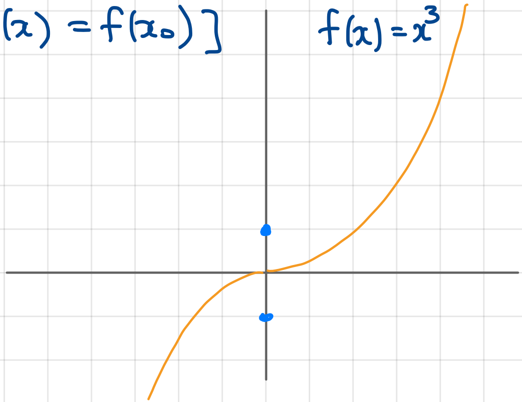
Week 7:

- Homework 6 (due Sunday, February 21)
- Midterm 2 (Wednesday, February 24): Lectures 8-16

Limit of a Function

Def 17.1 (Continuity). Let f be a real-valued function, $\mathrm{dom}(f) \subset \mathbb{R}$.
Function f is continuous at $x_0 \in \mathrm{dom}(f)$ if for any sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$

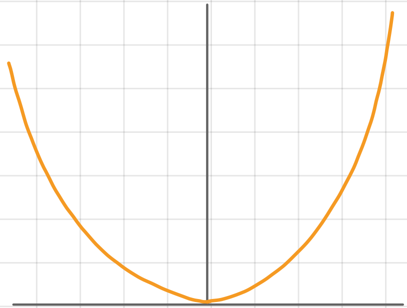
$$\lim f(x_n) = f(\lim x_n) \quad \left[\lim_{x \to x_0} f(x) = f(x_0) \right]$$



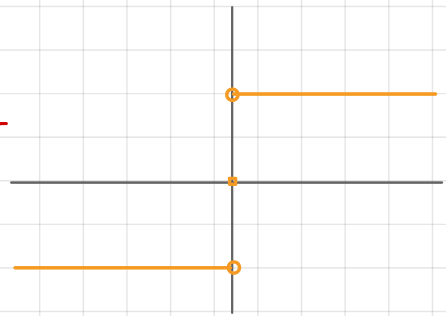
Def 20.1 (Limit of a function)
Let $S \subset \mathbb{R}$, $a, L \in \mathbb{R} \cup \{-\infty, +\infty\}$, suppose that there is a sequence in S for which a is the limit. Let $f: S \to \mathbb{R}$ be a function.
We say that f tends to L as x tends to a along S, or that L is the limit of f as x tends to a along S, if for every sequence (x_n) in S $\left(\lim x_n = a \Rightarrow \lim f(x_n) = L \right)$. Notation $\displaystyle\lim_{S \ni x \to a} f(x) = L$

Limit of a Function

Definitions 20.3



(a) We say that f tends to L as x tends to a, or that L is the (two-sided) limit of f as x tends to a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S = (a-c, a+c) \setminus \{a\}$ with $c > 0$; $\lim\limits_{x \to a} f(x) = L$



(b) L is the right-hand limit of f at a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S = (a, a+c)$ with $c > 0$; $\lim\limits_{x \to a^+} f(x) = L$

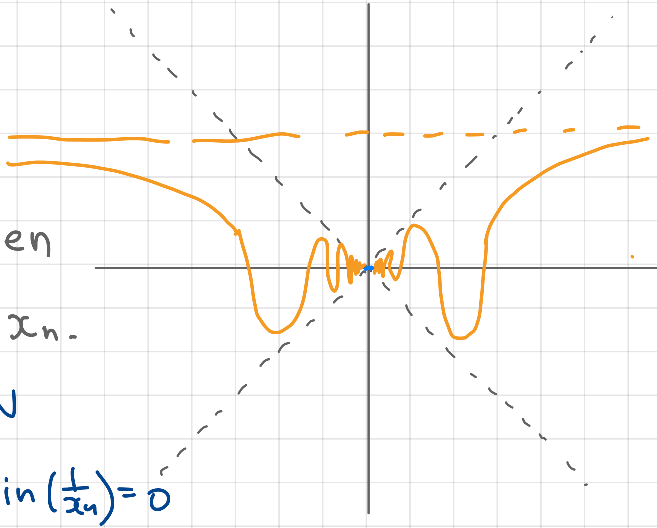
(c) L is the left-hand limit of f at a if $\lim\limits_{S \ni x \to a} f(x) = L$ for $S = (a-c, a)$ with $c > 0$; $\lim\limits_{x \to a^-} f(x) = L$

(d) $\lim\limits_{x \to +\infty} f(x) = L \iff \lim\limits_{S \ni x \to +\infty} f(x) = L$ for $S = (c, +\infty)$, $c \in \mathbb{R}$

$\lim\limits_{x \to -\infty} f(x) = L \iff \lim\limits_{S \ni x \to -\infty} f(x) = L$ for $S = (-\infty, c)$, $c \in \mathbb{R}$

Examples

1) $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$



Take any $c > 0$. Take any sequence (x_n) in $(-c, c) \setminus \{0\}$ s.t. $\lim x_n = 0$. Then $x \mapsto x \sin\left(\frac{1}{x}\right)$ is well-defined for all x_n.

Fix $\varepsilon > 0$, $\exists N \ \forall n > N \ |x_n| < \varepsilon \Rightarrow \forall n > N$

$$\left| x_n \cdot \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n| < \varepsilon \Rightarrow \lim x_n \sin\left(\frac{1}{x_n}\right) = 0$$

2) $\lim\limits_{x \to +\infty} x \sin\left(\frac{1}{x}\right) = 1$

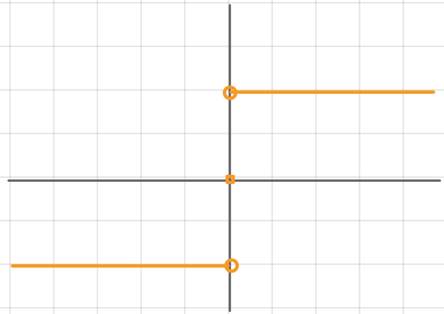
Take any $c > 0$. Take any sequence (x_n) in $(c, +\infty)$, $\lim x_n = +\infty$.

Denote $y_n = \frac{1}{x_n}$. Then by T.9.10 $\lim y_n = 0$

$$\forall n \quad x_n \sin\left(\frac{1}{x_n}\right) = \frac{\sin(y_n)}{y_n} \Rightarrow \lim x_n \sin\left(\frac{1}{x_n}\right) = \lim \frac{\sin(y_n)}{y_n} = 1$$

Examples

4) $f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$



$\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$: let (x_n) be a sequence, $x_n \in (0,1)$, $\lim x_n = 0$. Then

$$\forall n \quad |\operatorname{sgn}(x_n) - 1| = 0 \Rightarrow \lim_{x\to 0^+} \operatorname{sgn}(x) = 1$$

$\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist: Take a sequence $x_n = \frac{(-1)^n}{n}$

$\lim x_n = 0$, but $\operatorname{sgn}(x_n) = (-1)^n$, $((-1)^n)_{n=1}^{\infty}$ diverges.

5) $f(x) = \frac{x+1}{x-1}$, not defined at $x=1$

$\lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty$: take (x_n), $\lim x_n = 1$, $x_n > 1 \Rightarrow \frac{x_n+1}{x_n-1} > \frac{2}{x_n-1}$

Fix $M>0$, $\exists N$ $\forall n>N$ $|x_n - 1| = x_n - 1 < \frac{2}{M} \Rightarrow \forall n>N$ $\frac{2}{x_n-1} > M \Rightarrow \lim f(x_n) = +\infty$

6) If $f: S \to \mathbb{R}$ is continuous at $a \in S$, then $\lim_{S\ni x\to a} f(x) = f(a)$

$\frac{x+1}{x-1}$ is continuous at $x=-1 \Rightarrow \lim_{x\to -1} \frac{x+1}{x-1} = \frac{-1+1}{-1-2} = 0$

Limits and arithmetic operations

Thm 20.4 Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{S \ni x \to a} f_1(x)$ and $L_2 = \lim_{S \ni x \to a} f_2(x)$ exist and are finite. Then

(i) $\lim_{S \ni x \to a} (f_1 + f_2)(x) = L_1 + L_2$ (ii) $\lim_{S \ni x \to a} (f_1 f_2)(x) = L_1 \cdot L_2$

(iii) if $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$, then $\lim_{S \ni x \to a} \frac{f_1}{f_2}(x) = \frac{L_1}{L_2}$

Proof. Follows from Thm. 9.3, 9.4, 9.6.

Take any sequence (x_n) in S that converges to a. Then

$$\lim f_1(x_n) = L_1, \quad \lim f_2(x_n) = L_2.$$ Then

(i) By Thm 9.3 $\lim(f_1(x_n) + f_2(x_n)) = \lim f_1(x_n) + \lim f_2(x_n) = L_1 + L_2$

(ii) By Thm 9.4 $\lim(f_1(x_n) \cdot f_2(x_n)) = \lim f_1(x_n) \cdot \lim f_2(x_n) = L_1 \cdot L_2$

(iii) By Thm 9.6 $\lim \frac{f_1(x_n)}{f_2(x_n)} = \frac{\lim f_1(x_n)}{\lim f_2(x_n)} = \frac{L_1}{L_2}$ ∎

Limit of a composition of functions

Thm 20.5

(a) $\lim\limits_{S \ni x \to a} f(x) = L$

(b) g is defined on $\{f(x) : x \in S\} \cup \{L\}$

(c) g is continuous at L

$\Rightarrow \lim\limits_{S \ni x \to a} g \circ f(x) = g(L)$

Proof Let (x_n) be a sequence in S, $\lim x_n = a$.

(a) $\Rightarrow$ $\lim f(x_n) = L$

(b)+(c) $\Rightarrow$ $\lim g \circ f(x_n) = \lim g(f(x_n)) = g(L)$

Example

$f(x) = \sin(x)$, $g(x) = \operatorname{sgn}(x)$ – not continuous at 0. Then

for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad g \circ f(x) = \operatorname{sgn}(\sin(x)) = \operatorname{sgn}(x)$ – no limit at 0

Important example II

(A) Let $a>1$. Then $\lim_{x\to 0} a^x = 1 = a^0$ ($x \mapsto a^x$ is continuous at 0)

Take any sequence (x_n) in $\mathbb{R}\setminus\{0\}$, $\lim x_n = 0$. Fix $\varepsilon>0$.

① By IE 4 $\lim_{m\to\infty} a^{\frac{1}{m}} = 1 \Rightarrow \exists M_1\ \forall m>M_1\ \ a^{\frac{1}{m}} - 1 < \varepsilon$

② By IE 4 and Thm 9.5 $\lim_{m\to\infty} a^{-\frac{1}{m}} = \lim_{m\to\infty} \frac{1}{a^{\frac{1}{m}}} = 1 \Rightarrow \exists M_2\ \forall m>M_2\ 1-a^{-\frac{1}{m}} < \varepsilon$

③ Take $m>\max\{M_1, M_2\}$; $\lim x_n = 0 \Rightarrow \exists N\ \forall n>N\ \left(-\frac{1}{m} < x_n < \frac{1}{m}\right)$

④ $\forall n>N\ \left(a^{-\frac{1}{m}} < a^{x_n} < a^{\frac{1}{m}}\right)$

$\Rightarrow \forall n>N\ \left(-\varepsilon < a^{-\frac{1}{m}} - 1 < a^{x_n} - 1 < a^{\frac{1}{m}} - 1 < \varepsilon\right) \Rightarrow \lim a^{x_n} = 1 = a^0$

(B) Let $a>1$. Then $x \mapsto a^x$ is continuous on $\mathbb{R}$. Take $x_0 \in \mathbb{R}$, take (x_n), $x_n \neq x_0$, $\lim x_n = x_0$. Then $\lim a^{x_n} = \lim a^{x_0}\cdot a^{x_n - x_0} = a^{x_0} \lim a^{x_n - x_0}$

$\lim a^{x_n - x_0} = a^{x_0}$

(By (A) + $\lim(x_n - x_0) = 0 \Rightarrow$

Important example II

(C) $\forall a > 0$, $x \mapsto a^x$ is continuous on $\mathbb{R}$

If $a \in (0,1)$, then $\forall x \in \mathbb{R}$ $a^x = \left(\frac{1}{b}\right)^x = b^{-x}$, where $b = \frac{1}{a} > 1$

$g(x) = b^x$ is continuous by (B), $f(x) = -x$ is continuous by Thm 17.3

composition $g \circ f(x)$ is continuous (on $\mathbb{R}$) by Thm 17.5

If $a = 1$, then $a^x = 1$ $\forall x$, continuous.

(D) $\forall a > 0, a \neq 1$, $x \mapsto \log_a x$ is continuos on $(0, +\infty)$ by Thm 18.4

$x \to a^x$ is strictly increasing $(a > 1)$ or strictly decreasing $(a < 1)$

and maps $\mathbb{R}$ to $(0, +\infty)$