# MATH 142A: Introduction to Analysis

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## Today: Continuous functions

> Q&A: February 8

## Next: Ross § 18

Week 6:

- Homework 5 (due Sunday, February 14)
- Regrades of HW3 (Monday, February 8 - Wednesday, February 10)

# Functions

<u>Def. (Function)</u> Let $X$ and $Y$ be two sets. We say that there is a function defined on $X$ with values in $Y$, if via some rule $f$ we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f: X \to Y$, $x \overset{f}{\mapsto} y$ (or $y = f(x)$).

$X$ is called the domain of definition of the function, $\mathrm{dom}(f)$,

$y = f(x)$ is called the image of $x$. $f: [0,1) \to [0,1)$, $x \mapsto x^2$

<u>Remarks</u> 1) We consider real-valued functions ($Y \subset \mathbb{R}$) of one real variable ($X \subset \mathbb{R}$).

2) If $\mathrm{dom}(f)$ is not specified, then it is understood that we take the natural domain: the largest subset of $\mathbb{R}$ which the function is well defined

$$\left( \begin{array}{l} f(x) = \sqrt{x} \text{ means } \mathrm{dom}(f) = [0, +\infty) \\ g(x) = \dfrac{1}{x^2 - x} \text{ means } \mathrm{dom}(g) = \mathbb{R} \setminus \{0, 1\} \end{array} \right)$$

## Continuity of a function at a point

Intuitively : Function $f$ is continuous at point $x_0 \in \mathrm{dom}(f)$ if

$f(x)$ approaches $f(x_0)$ as $x$ approaches $x_0$.

Def 17.1 (Continuity). Let $f$ be a real-valued function, $\mathrm{dom}(f) \subset \mathbb{R}$.

Function $f$ is continuous at $x_0 \in \mathrm{dom}(f)$ if for any sequence

$(x_n)$ in $\mathrm{dom}(f)$ converging to $x_0$, we have

Def 17.6 (Continuity) Let $f$ be a real-valued function.

Function $f$ is continuous at $x_0 \in \mathrm{dom}(f)$ if

Remark Def 17.1 is called the sequential definition of continuity,

Def 17.6 is called the $\varepsilon$-$\delta$ definition of continuity.

# Equivalence of sequential and $\varepsilon$-$\delta$ definitions

**Thm 17.2**. Definitions 17.1 and 17.6 are

**Proof** $(17.1 \Rightarrow 17.6)$. Suppose that $(*)$ fails

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \left( x \in \mathrm{dom}(f) \wedge |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon \right) \quad (*)$$

This means that

Take $\delta =$ :

$\Rightarrow$

$(\Leftarrow)$. Let $(x_n)$ be such that $\lim x_n = x_0$. Fix $\varepsilon>0$. By $(*)$

$\lim\limits_{n\to\infty} x_n = x_0 \Rightarrow$ Therefore

$\forall n>N \ (x_n \in \mathrm{dom}(f) \wedge |x_n-x_0|<\delta) \overset{(*)}{\Longrightarrow}$

# Continuity on a set. Examples

**Def 17.1** Let $f$ be a function, and let $S \subset \mathrm{dom}(f)$.

$f$ is continuous on $S$ if for all $x_0 \in S$ $f$ is continuous at $x_0$.

**Example** 1) $f(x) = \frac{2x}{x^2-1}$ is continuous on $\mathbb{R} \setminus \{-1,1\}$

**Proof.** Let $x_0 \in \mathbb{R} \setminus \{-1,1\}$ and let $(x_n)$ be such that $\forall n$ $x_n \notin \{-1,1\}$

and $\lim x_n = x_0$. Then by Thm 9.2, 9.3, 9.6

$\lim f(x_n) =$

By Def 17.1 $f$ is continuous at $x_0$ for any $x_0 \in \mathbb{R} \setminus \{-1,1\}$

2) $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = a$. Then for any $a \in \mathbb{R}$

$g$ is not continuous at $0$.

**Proof** Take $(x_n)$ with $x_n =$

Then and

$\Rightarrow$

# Continuity and arithmetic operations

**Thm 17.3** Let $f$ be a real-valued function with $\mathrm{dom}(f) \subset \mathbb{R}$.

If $f$ is continuous at $x_0 \in \mathrm{dom}(f)$, then

**Proof.** Let $(x_n)$ be a sequence in $\mathrm{dom}(f)$ such that $\lim_{n\to\infty} x_n = x_0$.

Then by Thm 9.2

Therefore $k \cdot f$ is continuous at $x_0$.

By the triangle inequality

Fix $\varepsilon > 0$. Then $\lim f(x_n) = f(x_0) \Rightarrow$

Then $\forall\, n > N$

This means that $\lim_{n\to\infty} |f(x_n)| = |f(x_0)|$, $|f|$ is continuous at $x_0$.

# Continuity and arithmetic operations

<u>Thm 17.4</u> Let $f$ and $g$ be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then

(i) $f+g$ is continuous at $x_0$ (ii) $f \cdot g$ is continuous at $x_0$

(iii) if $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at $x_0$.

<u>Proof</u>: Note that if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then $(f+g)(x) = f(x) + g(x)$ and $f \cdot g(x) = f(x) \cdot g(x)$ are well-defined. Moreover, if $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is well-defined.

Let $(x_n)$ be a sequence in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ s.t. $\lim x_n = x_0$.

Then $\lim (f(x_n) + g(x_n)) =$ , and

$\lim (f(x_n) \cdot g(x_n)) =$ . If moreover $\forall n\ g(x_n) \neq 0$

then $\lim \frac{f(x_n)}{g(x_n)} =$

# Continuity of a composition of functions

Let $f$ and $g$ be real-valued functions. If $x \in dom(f)$ and $f(x) \in dom(g)$,

then we define

<u>Thm 17.5</u> If $f$ is continuous at $x_0$ and $g$ is continuous at $f(x_0)$,

then

<u>Proof</u> It is given that $x_0 \in dom(f)$ and $f(x_0) \in dom(g)$.

Let $(x_n)$ be a sequence such that and

$\lim x_n = x_0$. Denote . Since $f$ is continuous

at $x_0$, $\lim y_n =$ Since $g$ is continuous

at $f(x_0) = y_0$, we have $\lim g \circ f(x_n) =$

Therefore, $g \circ f$ is continuous at $x_0$.

## Examples

1) $\sin(x)$ is continuous on $\mathbb{R}$

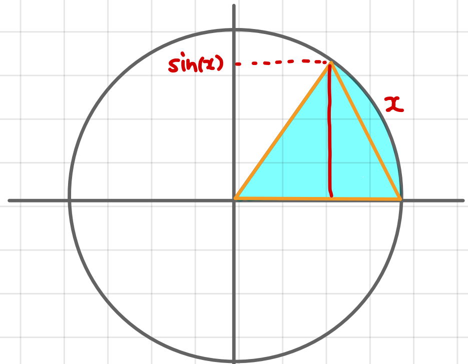
Proof ① Enough to show that $\sin(x)$ is continuous at $0$

For any $x_0 \in \mathbb{R}$ and $(x_n)$ with $\lim x_n = x_0$

$$|\sin(x_n) - \sin(x_0)| =$$

② Area ($\triangle$) $\leq$ Area ($\triangle$)

$\Rightarrow \forall\ x \in [0, \frac{\pi}{2}]$




$\Rightarrow$

③ If $\lim y_n = 0$, then

$\forall\ n > N$

Then

## Examples

2) $f(x) = \sqrt{x}$ is continuous on $[0, +\infty)$.

① $\sqrt{x}$ is continuous at $0$

Let $\lim x_n = 0$. Fix $\varepsilon > 0$. Then

$\Rightarrow$

② Let $x_0 \in (0, +\infty)$, $(x_n)$ s.t. $\forall n \, (x_n \in [0, +\infty))$ and $\lim x_n = x_0$

Then $\lim x_n = x_0 > 0 \overset{T9.11(i)}{\Rightarrow}$

Fix $\varepsilon > 0$. Then . Then

$\forall n > \max\{N_1, N_2\} \; |f(x_n) - f(x_0)| = |\sqrt{x_n} - \sqrt{x_0}| =$

3) $\cos(x)$ is continuous on $\mathbb{R}$. $\cos(x) =$ , by Thm 17.4

is continuous on $\mathbb{R}$. Moreover, $\forall x \in \mathbb{R}$

$\Rightarrow$ by example 2) and Thm 17.5