MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Series

> Q&A: February 5

Next: Ross § 17

Week 5:

- Homework 4 (due Sunday, February 7)

Comparison test

Thm 14.6 Let (a_n) and (b_n) be two sequence, $\forall n\ a_n \geq 0$

Then

(i) $\left(\sum_{n=1}^{\infty} a_n \text{ converges} \wedge \forall n \left(|b_n| \leq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n \text{ converges}$

(ii) $\left(\sum_{n=1}^{\infty} a_n = +\infty \wedge \forall n \left(b_n \geq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$

Examples

-

-

Corollary 14.7 Absolutely convergent series are convergent

Proof:

Root Test

Thm 14.9 Let $\sum_{n=1}^{\infty} a_n$ be a series, let $\alpha = \limsup \sqrt[n]{|a_n|}$. Then

(i) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$

(ii) $\alpha > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$

(iii) $\alpha = 1$ does not provide information about the convergence of $\sum_{n=1}^{\infty} a_n$

Proof: (i) $\alpha < 1 \Rightarrow \exists$

$\limsup \sqrt[n]{|a_n|} = \alpha$

$\Rightarrow$

Fix $\varepsilon > 0$. Since $\beta < 1$,

Then

(ii) $\exists\ (n_k)$ s.t.

Ratio Test

Thm 14.8 Let $\sum_{n=1}^{\infty} a_n$ be a series, $\forall n \, (a_n \neq 0)$.

(i) $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n$

(ii) $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum a_n$

(iii) $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$: not enough information.

Proof Let $\alpha = \limsup \sqrt[n]{|a_n|}$. Then by Thm 12.2

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \limsup \sqrt[n]{|a_n|} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

(i)

(ii)

(iii)

Examples

- $\forall \quad \alpha > 1$

  Ratio test:

  $\Rightarrow$

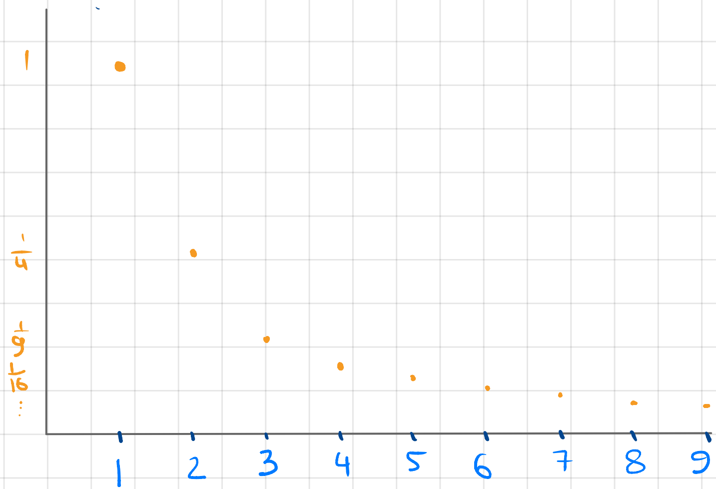
  - $\sum_{n=1}^{\infty} \frac{1}{n^2}$
  - $\sum_{n=1}^{\infty} \frac{1}{n}$

Ratio test:

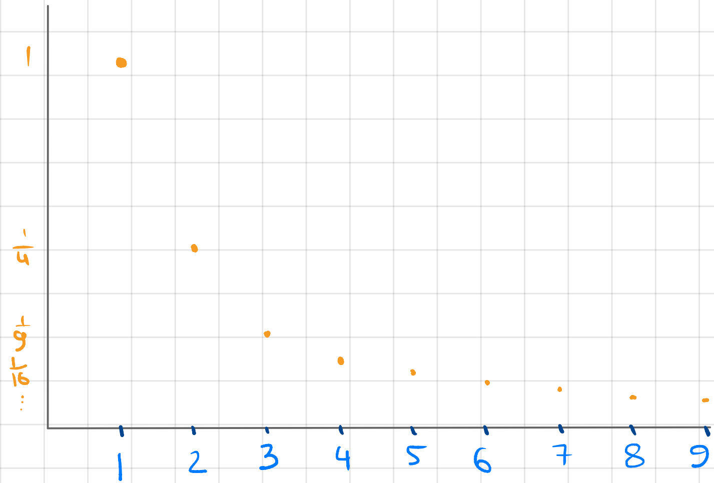
Cauchy test:

Integral test

- $a_n = \frac{1}{n^2}$,



- $b_n = \frac{1}{n}$,



- $p > 0$:

Examples

- $a_n = \frac{1}{n}$, $n \geq 3$ [use $\forall n \geq 3$ $1 \leq \log n \leq n$]

Root test:

Alternating Series

Thm 15.3 Let (a_n) be a sequence s.t. $\forall n\ (a_n \geq 0 \wedge a_n \geq a_{n+1})$. Then

$\lim_{n\to\infty} a_n = 0 \Rightarrow$

Proof. Denote $\sum_{k=1}^{\infty} a_k =: s$, $\sum_{k=1}^{n} a_k =: s_n$.

① $(s_{2n})_{n=1}^{\infty}$ is , $(s_{2n-1})_{n=1}^{\infty}$ is

② $\forall m, n \in \mathbb{N}$

Case $m \leq n$:

Case $m \geq n$:

By ② + Thm 10.2 and

Then $\forall n\ (s_{2n} \leq s \leq s_{2n+1}) \Rightarrow$

Important example

9. Let $p>0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$

Proof. Denote $x_n = \frac{1}{n^p}$, $S_k = \sum_{n=1}^{k} x_n$. $x_1 \geq x_2 \geq \cdots \geq x_n$, (S_k) is increasing.

Consider the sequences:

Then

and $\forall k$

① (S_k) converges $\Longleftrightarrow$

② (S_{2^k}) converges $\Longleftrightarrow$

$a_n =$