

MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Natural, rational, algebraic
numbers > Q&A: January 6

Next: Ross § 3

Week 1:

- visit course website
- homework 0 (due Friday, January 8)
- join Piazza

Logical symbolism

Common logical connectives

\neg negation ("not")

\wedge and

\vee or

\Rightarrow implies

\Leftrightarrow is equivalent to

Example:

A: Alice plays accordion

B: Bob reads a book

C: Alice and Bob stay at home

$$C \wedge A \Rightarrow \neg B$$

$$(A \wedge \neg B) \vee (\neg A \wedge B)$$

Typical mathematical statement: $A \Rightarrow B$

Typical proof: $(A \Rightarrow C_1) \wedge (C_1 \Rightarrow C_2) \wedge \dots \wedge (C_n \Rightarrow B)$

$$A \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \dots \Rightarrow C_n \Rightarrow B$$

Logical symbolism

Basic rules for constructing proofs

- if A is true and $A \Rightarrow B$, then B is true
- the law of excluded middle: $A \vee \neg A$ is always true
↳ used in proofs by contradiction
- rule of double negation: $\neg(\neg A) \Leftrightarrow A$

Use words instead of symbols (most of the time)

$A \Rightarrow B$

A implies B

B follows from A

B is necessary condition for A

A is sufficient condition for B

$A \Leftrightarrow B$

A is equivalent to B

A if and only if B

A is necessary and sufficient
for B

Logical symbolism

Think about the following statements

$$(a) \neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

$$(b) \neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

$$(c) (A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

Set theory notation

A set is a "collection of distinguishable objects"

- a set may consist of any distinguishable objects
- a set is uniquely determined by the collection of objects it consists of
- a set can be defined as a collection of objects having certain property

$\{x, y, z\}$ - listing objects

$\{x: P(x)\}$ - the set of all objects x that satisfy property P

$\{1, 2, 3, 4, 5\}$ or $\{n: n \in \mathbb{N} \text{ and } n \leq 5\}$

If S is a set, $x \in S$ means that x is an element of S

$x \notin S$ x is not an element of S

Set theory notation

S, T are two sets, then $T \subset S$ means that

each element of T belongs to S .

$$\{1, 2, 3, 4, 5\} \subset \mathbb{N}, \quad \mathbb{N} \subset \mathbb{R}, \quad \mathbb{R} \subset \mathbb{R}$$

$$\{\{1, 2\}, \{1, 3\}\}$$

Defining a set from another set by specifying a rule

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

Operations on sets

If we have 2 sets S, T , then

- $S \setminus T = \{x : (x \in S) \wedge (x \notin T)\}$ is the difference between S and T
- $S \cup T = \{x : (x \in S) \vee (x \in T)\}$ is the union of S and T
- $S \cap T = \{x : (x \in S) \wedge (x \in T)\}$ is the intersection of S and T

Set theory notation

\mathcal{A} is a set, $S_\alpha, \alpha \in \mathcal{A}$, is a collection of sets, then

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for at least one } \alpha \in \mathcal{A}\}$$

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in \mathcal{A}\}$$

Examples $S = \{1, 2, 4, 6\}$, $T = \{1, 3, 5\}$

$$S \setminus T = \{2, 4, 6\}$$

$$S \cup T = \{1, 2, 3, 4, 5, 6\}$$

$$S \cap T = \{1\}$$

Empty set is the set with no elements, \emptyset

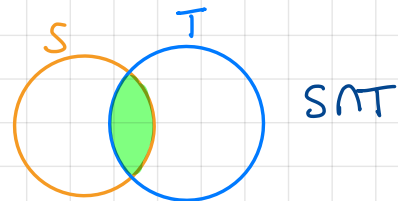
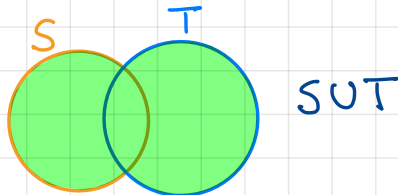
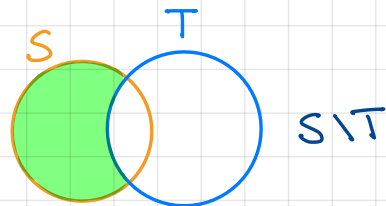
$$S = \{1, 2, 3\}$$

$$S \cap T = \emptyset$$

$$T = \{4, 5, 6\}$$



$$S \cap T = \emptyset$$



Natural numbers

We assume that we know what natural numbers are:
numbers we use to count objects.

$$\{1, 2, 3, 4, \dots\} =: \mathbb{N}$$

Peano Axioms:

$$N1. 1 \in \mathbb{N}$$

$$N2. n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$$

successor

N3. For any $n \in \mathbb{N}$, $n+1=1$ is false

$$N4. (m, n \in \mathbb{N}) \wedge (m+1=n+1) \Rightarrow m=n$$

$$N5. S \subset \mathbb{N} \wedge 1 \in S \wedge (n \in S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$$

Properties N1 - N5 define \mathbb{N} uniquely.

Principle of mathematical induction

Let P_1, P_2, P_3, \dots be a list of statements that may or may not be true. Then

(I₁) P_1 is true
(I₂) P_n is true $\Rightarrow P_{n+1}$ is true

$\Bigg| \Rightarrow$ all statements P_1, P_2, P_3, \dots
are true

(I₁) basis of induction (I₂) induction step

$$N5. \quad S \subset \mathbb{N} \wedge 1 \in S \wedge (n \in S \Rightarrow n+1 \in S) \Rightarrow S = \mathbb{N}$$

Suppose that (I₁) and (I₂) hold. Define

$$S := \{n \in \mathbb{N} : P_n \text{ is true}\}$$

(I₁) $\Rightarrow 1 \in S$
(I₂) $\Rightarrow (n \in S \Rightarrow n+1 \in S)$

$\Bigg| \xRightarrow{N5} S = \mathbb{N} \Leftrightarrow$ all statements P_1, P_2, \dots
are true

Example

Prove that for real $x > -1$ and for any $n \in \mathbb{N}$

$$(1+x)^n \geq 1+nx$$

Solution: Fix $x > -1$. Denote P_n : " $(1+x)^n \geq 1+nx$ ".

- P_1 : $1+x \geq 1+x$ is true (basis of induction)
- Suppose that P_n is true, i.e., $(1+x)^n \geq 1+nx$. Then
$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx + x + nx^2 \geq 1+(n+1)x, \text{ i.e.}$$
$$P_n \Rightarrow P_{n+1}.$$
 Induction step holds. By the principle of mathematical induction, P_n is true for all $n \in \mathbb{N}$.

Remark

Principle of mathematical induction with different basis

Let P_1, P_2, P_3, \dots be a list of statements that may or may not be true. Let $k \in \mathbb{N}$. Then

(I₁') P_k is true
(I₂') P_n is true $\Rightarrow P_{n+1}$ is true
for all $n \geq k$

\Rightarrow all statements $P_k, P_{k+1}, P_{k+2}, \dots$ are true

Proof. Define $P'_n = P_{n+k-1}$, $n \in \mathbb{N}$, and apply the principle of mathematical induction for P'_1, P'_2, P'_3, \dots ■

Example Prove that for all $n \in \mathbb{N}$, $n \geq 2$ $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Solution. $P_1: 1 > \sqrt{1}$ is false. $P_2: 1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true. For $n \geq 2$ if $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$, then $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n^2+n} + 1}{\sqrt{n+1}} > \sqrt{n+1}$
induction step is true. Principle of math. induction implies the result.

Integer and rational numbers

$\mathbb{Z} := \mathbb{N} \cup \{n : n \in \mathbb{N}\} \cup \{0\} = \{0, 1, -1, 2, -2, \dots\}$ integer numbers

$$\frac{1}{2} \notin \mathbb{Z}$$

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$\mathbb{Q} := \left\{ \frac{m}{n} : (m, n \in \mathbb{Z}) \wedge (n \neq 0) \right\}$ rational numbers

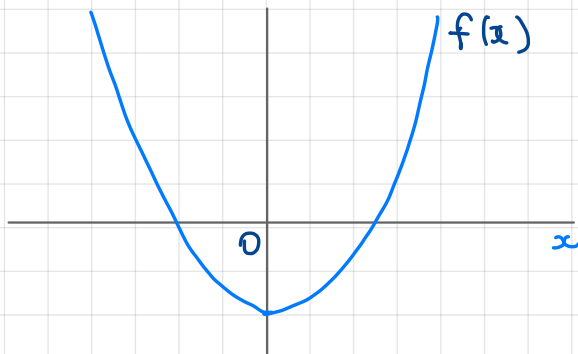
$\mathbb{Q} \setminus \{0\}$ is closed with respect to four arithmetic operations

Are there any other numbers?

Consider polynomial equation $x^2 = 2$

$$(\pm\sqrt{2})^2 - 2 = 0 \quad \sqrt{2} \notin \mathbb{Q}$$

$$f(x) = x^2 - 2$$



Algebraic numbers

Definition 2.1 (Algebraic number)

A number is called algebraic if it satisfies a polynomial equation $C_n x^n + C_{n-1} x^{n-1} + \dots + C_1 x + C_0 = 0$, where C_0, \dots, C_n are integers and $n \geq 1$.

Remark Rational numbers are algebraic numbers: for $q = \frac{k}{l}$

take $n=1$, $C_0 = -k$, $C_1 = l$, giving the equation $lx - k = 0$

Examples of algebraic numbers:

$$\sqrt[4]{17}$$

$$x^4 = 17$$

$$\sqrt{2}$$

$$x^2 = 2$$

$$\sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

$$((x^2 - 2)^2 - 2)^2 - 2 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2 = 0$$

$\sqrt{2} \notin \mathbb{Q}$

Theorem 2.2 (Rational Zeros Theorem)

Suppose that c_0, c_1, \dots, c_n are integers and r is a rational number satisfying the polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 = 0 \quad (*)$$

Let $r = \frac{c}{d}$ where c and d are integers having no common factors.

Then c divides c_0 and d divides c_n .

$$\left(\frac{c}{d} \text{ solves } (*) \Rightarrow c \text{ divides } c_0, d \text{ divides } c_n \right)$$

Proof. No proof. ■

Corollary. If r satisfies $r^2 - 2 = 0$, then $r \notin \mathbb{Q}$.

Proof. Let r be such that $r^2 - 2 = 0$. If $r \in \mathbb{Q}$, then by Thm 2.2.

$r \in \{1, 2, -1, -2\}$. If $r \in \{1, 2, -1, -2\}$, then $r^2 - 2 \in \{-1, 2\}$, $r^2 - 2 \neq 0$.

Contradiction, $r \notin \mathbb{Q}$ ■