MATH 142A: Introduction to Analysis

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Today: Series

> Q&A: February 3

Next: Ross § 15

Week 5:

- Homework 4 (due Sunday, February 7)
- Quiz 3 (Wednesday, February 3) - Lectures 8-9

Sequences $\left|\frac{s_{n+1}}{s_n}\right|$ and $\sqrt[n]{|s_n|}$

Thm 12.2 Let (s_n) be a sequence, $\forall n\ (s_n \neq 0)$. Then

$$\underbrace{\liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|}_{=:\ell} \le \underbrace{\liminf_{n\to\infty}\sqrt[n]{|s_n|}}_{=:\beta} \le \limsup_{n\to\infty}\sqrt[n]{|s_n|} \le \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$$

Proof. If $\ell = 0$, then $\ell \le \beta$. Assume that $\ell > 0$.

Take any $0<\ell_1<\ell$. Then by Thm 9.11 (i) $\exists N$ s.t.

$$\inf\left\{\left|\frac{s_{n+1}}{s_n}\right| : n\ge N\right\} > \ell_1 \implies \forall n \ge N \quad \left|\frac{s_{n+1}}{s_n}\right| > \ell_1 .$$

Therefore, $\forall n > N$

$$|s_n| = |s_N|\cdot\frac{|s_{N+1}|}{|s_N|}\cdot\frac{|s_{N+2}|}{|s_{N+1}|}\cdot\ldots\cdot\frac{|s_n|}{|s_{n-1}|} > |s_N|\cdot\ell_1^{\,n-N} = \frac{|s_N|}{\ell_1^N}\cdot\ell_1^n$$

$$\implies \forall n>N \quad \sqrt[n]{|s_n|} > \sqrt[n]{\frac{|s_N|}{\ell_1^N}\cdot\ell_1^n} = \ell_1\cdot\sqrt[n]{\frac{|s_N|}{\ell_1^N}} \implies \overbrace{\inf\left\{\sqrt[n]{|s_n|} : n>N\right\}}^{=:\tilde{u}_N} \ge \ell_1\sqrt[n]{\frac{|s_N|}{\ell_1^N}}$$

Note that $(\tilde{u}_k)$ is increasing, so $\forall k > N \quad \tilde{u}_k \ge \tilde{u}_N \ge \ell_1\sqrt[n]{\frac{|s_N|}{\ell_1^N}}$

Now $$\beta = \lim_{k\to\infty}\tilde{u}_k \overset{\text{Cor 9.12}}{\ge} \lim_{n\to\infty}\left(\ell_1\sqrt[n]{\frac{|s_N|}{\ell_1^N}}\right) = \ell_1\lim_{n\to\infty}\sqrt[n]{\frac{|s_N|}{\ell_1^N}} \overset{\text{IE4}}{=} \ell_1\cdot 1 = \ell_1$$

So $\forall \ell_1\in(0,\ell)\ (\beta\ge\ell_1) \implies \beta$ is an upper bound for $(0,\ell) \implies \beta \ge \ell$ ∎

Sequences $\left|\frac{s_{n+1}}{s_n}\right|$ and $\sqrt[n]{|s_n|}$

Corollary 12.3

If $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ exists, and $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = L$, then $\lim_{n\to\infty}\sqrt[n]{|s_n|} = L$

Example

Let (a_n) be a sequence such that $\forall n\in\mathbb{N}$ $a_n > 0$.

Suppose that (a_n) converges, $\lim_{n\to\infty} a_n = a$. Then

$$\lim_{n\to\infty}\sqrt[n]{a_1\cdot a_2\cdot\dots a_n} = a$$

Proof. Denote $s_n := a_1\cdot\dots\cdot a_n$. Then $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty} a_{n+1} = a$

By Corollary 12.3 $\lim_{n\to\infty}\sqrt[n]{|s_n|} = a = \lim_{n\to\infty}\sqrt[n]{a_1\cdots a_n}$ ∎

Series

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

For $p, q \in \mathbb{N}$, $p<q$ we denote $a_p + a_{p+1} + \cdots + a_q$ by $\sum_{n=p}^{q} a_n$

Def 14.1 (Infinite series) We call the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots =: \sum_{n=1}^{\infty} a_n$$

an (infinite) series. a_n is called the n-th term of the series.

Def 14.2 (Convergent series)

We call the sum $S_n = \sum_{k=1}^{n} a_k$ the (n-th) partial sum of the series.

If the sequence (S_n) of partial sums converges, we say that the series $\sum_{n=1}^{\infty} a_n$ is convergent

If $\lim_{n\to\infty} S_n = S$, then we call S the sum of the series $\sum_{n\geq 1} a_n$, and write it as $\sum_{n=1}^{\infty} a_n = S$

Series

If $\lim_{n\to\infty} S_n = +\infty\ (-\infty)$, we say that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty\ (-\infty)$

and we write $\sum_{n=1}^{\infty} a_n = +\infty$ (or $-\infty$)

We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent)

if the series $\sum_{n=1}^{\infty} |a_n|$ converges

<u>Remark</u> An infinite series can be viewed as a particular type of a sequence, $S_n = a_1 + a_2 + \cdots + a_n$

so we can use all the relevant results.

For example, if $\forall n\ a_n \geq 0$, then S_n is increasing.

Partial sums of $\sum_{n=1}^{\infty} |a_n|$ form an increasing sequence.

Use the criteria on convergence for partial sums etc.

Important examples

8. Let $a, r \in \mathbb{R}$. Then $\sum_{n=0}^{\infty} ar^n$ is called the geometric series.

If $|r|<1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Proof Denote $S_k = \sum_{n=0}^{k} ar^n = a + ar + \cdots + ar^k = a(1+r+r^2+\cdots+r^k)$

Note that $r(1+r+\cdots+r^k) = r + r^2 + \cdots + r^k + r^{k+1}$, so

$$(1-r)(1+r+\cdots+r^k) = 1+r+\cdots+r^k - r(1+r+\cdots+r^k) = 1-r^{k+1} \Rightarrow 1+r+\cdots+r^k = \frac{1-r^{k+1}}{1-r}$$

$\Rightarrow \forall k \quad S_k = \frac{a(1-r^{k+1})}{1-r}$. By 1.E2 $\lim_{k\to\infty} r^{k+1} = 0 \Rightarrow \lim_{k\to\infty} S_k = \frac{a}{1-r}$ ∎

9. Let $p>0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$

Proof $(p=2)$. $S_k := \sum_{n=1}^{k} \frac{1}{n^2}$. ① (S_k) is increasing $\left(S_{k+1} - S_k = \frac{1}{(k+1)^2} > 0\right)$

② (S_k) is bounded $S_k = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k-1)k}$

$\boxed{\frac{\pi^2}{6}}$

For any $n \ge 2$ $\frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$, so $S_k \le 1 + \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\cdots} + \cancel{\frac{1}{k-1}} - \frac{1}{k} = 2 - \frac{1}{k} < 2$

① + ② + Thm 10.2 ∎

Cauchy criterion

Def 14.3 We say that $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion if its sequence of partial sums (S_n) is a Cauchy sequence, i.e.

$$\forall \varepsilon>0 \ \exists N \ \forall m,n>N \ |S_n-S_m|<\varepsilon$$

$$\forall \varepsilon>0 \ \exists N \ \forall n>m>N \ \left|\sum_{k=m+1}^{n} a_k\right|<\varepsilon$$

Thm 14.4 $\sum a_n$ converges $\iff$ $\sum a_n$ satisfies the Cauchy criterion

Proof. Follows from Thm 10.11

Corollary 14.5 (Necessary condition for convergence).

$$\sum a_n \text{ converges} \Rightarrow \lim_{n\to\infty} a_n = 0$$

Proof. $\sum a_n$ converges $\overset{\text{Thm 14.4}}{\iff} \forall \varepsilon>0 \ \exists N \ \forall n>m>N \ \left|\sum_{k=m+1}^{n} a_k\right|<\varepsilon$ (take $n=m+1$)

$\Rightarrow \forall \varepsilon>0 \ \exists N \ \forall n>N+1 \ \left(|a_n|<\varepsilon\right) \iff \lim a_n = 0$ ∎

Example

- $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ satisfies the Cauchy criterion

Proof. $\forall\, k \in \mathbb{N}\ \frac{1}{k \cdot 2^k} \le \frac{1}{2^k}$, so $\forall\, n > m \ge 1$

$$\sum_{k=m+1}^{n} \frac{1}{k2^k} \le \sum_{k=m+1}^{n} \frac{1}{2^k} = \frac{1}{2^{m+1}} \sum_{l=0}^{n-m} \frac{1}{2^l} < \frac{1}{2^{m+1}} \sum_{l=0}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m+1}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^m}$$

Fix $\varepsilon > 0$. By 1.E.2 $\exists N\ \forall m > N \left(\frac{1}{2^m} < \varepsilon\right)$. Therefore

$\forall\, n > m > N \quad \sum_{k=m+1}^{n} \frac{1}{k2^k} \le \frac{1}{2^m} < \varepsilon \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k2^k}$ satisfies the Cauchy criterion

In particular, $\lim_{k \to \infty} \frac{1}{k2^k} = 0$ $\blacksquare$

- If $|r| \ge 1$, then the sequence (r^n) does not converge to 0 (L8)

$\Rightarrow \sum_{n=0}^{\infty} r^n$ does not converge

- Consider $\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n \to \infty} \frac{1}{n} = 0$, but the series diverges (L8)

Comparison test

Thm 14.6 Let (a_n) and (b_n) be two sequence, $\forall n \; a_n \geq 0$

Then

(i) $\left(\sum_{n=1}^{\infty} a_n \text{ converges} \wedge \forall n \left(|b_n| \leq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n$ converges

(ii) $\left(\sum_{n=1}^{\infty} a_n = +\infty \wedge \forall n \left(b_n \geq a_n \right) \right) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty$

Proof. (i) Use the Cauchy criterion $\left[\forall \varepsilon > 0 \; \exists N \; \forall n > m > N \; \left| \sum_{k=m+1}^{n} b_k \right| < \varepsilon \right]$

$$\left| \sum_{k=m+1}^{n} b_k \right| \overset{\text{Tr. Ineq.}}{\leq} \sum_{k=m+1}^{n} |b_k| \leq \sum_{k=m+1}^{n} a_k$$

Fix $\varepsilon > 0$. By Thm 14.4 $\exists N \; \forall n > m > N \; \sum_{k=m+1}^{n} a_k < \varepsilon$. Then

$\forall n > m > N \; \left| \sum_{k=m+1}^{n} b_k \right| < \sum_{k=m+1}^{n} a_k < \varepsilon$. By Thm 14.4 $\sum_{n=1}^{\infty} b_n$ converges

(ii) Denote $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$. Then $\forall n \; (t_n \geq s_n)$

$$\sum_{n=1}^{\infty} a_n = +\infty \iff \lim_{n\to\infty} s_n = +\infty \Rightarrow \lim_{n\to\infty} t_n = +\infty \iff \sum_{n=1}^{\infty} b_n = +\infty \quad \blacksquare$$