Nash Equilibrium Problems of Polynomials

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Introduction

Suppose there are N players and the ith player's strategy vector is $x_i \in \mathbb{R}^{n_i}$. Denote

$$\mathbf{x} := (x_1, \ldots, x_N), \quad x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N).$$

The Nash equilibrium problem (NEP) is to find a vector $\mathbf{x}$ such that for each $i = 1, \ldots, N$, the x_i solves the following optimization problem, for given x_{-i} of other player's strategies.

$$\mathbf{F}_i(x_{-i}) : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ s.t. & g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i), \\ & g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i). \end{cases} \tag{1.1}$$

A solution of the NEP is called a Nash equilibrium (NE).

Notations

In the above

- $f_i(x_i, x_{-i})$ - The ith player's objective function.
- $g_{i,j}(x_i)$ - The ith player's constraining function.
- $\mathcal{E}_i$ - The labelling set of equality constraints.
- $\mathcal{I}_i$ - The labelling set of inequality constraints.
- n_i - The dimension of x_i.

Besides that, we let

- $n \,:=\, n_1 + \cdots + n_N$.
- $m_i \,:= \mathcal{E}_i \cup \mathcal{I}_i$.
- $X_i \,:=\, \{x_i \in \mathbb{R}^{n_i} :\, g_{i,j}(x_i) = 0\,(j \in \mathcal{E}_i),\, g_{i,j}(x_i) \geq 0\,(j \in \mathcal{I}_i)\}$.
- $[k] \,:=\, \{1, \ldots, k\}$.

Introduction

Example 1.1

Consider the 2-player NEP with the individual optimization

$$\text{1st player:} \begin{cases} \min\limits_{x_1 \in \mathbb{R}^2} & x_{1,1}(x_{1,1} + x_{2,1} + 4x_{2,2}) + 2x_{1,2}^2, \\ s.t. & 1 - (x_{1,1})^2 - (x_{1,2})^2 \geq 0, \end{cases}$$

$$\text{2nd player:} \begin{cases} \min\limits_{x_2 \in \mathbb{R}^2} & x_{2,1}(x_{1,1} + 2x_{1,2} + x_{2,1}) \\ & \qquad + x_{2,2}(2x_{1,1} + x_{1,2} + x_{2,2}), \\ s.t. & 1 - (x_{2,1})^2 - (x_{2,2})^2 \geq 0. \end{cases}$$

Introduction

This NEP has only 3 NEs, which are

$$\begin{array}{rll} \text{1st NE:} & x_1^* = (0,0), & x_2^* = (0,0); \\ \text{2nd NE:} & x_1^* = (1,0), & x_2^* = \frac{1}{\sqrt{5}}(-1,-2); \\ \text{3rd NE:} & x_1^* = (-1,0), & x_2^* = \frac{1}{\sqrt{5}}(1,2). \end{array}$$

It is interesting to note that each player's objective is strictly convex with respect to its strategy, because their Hessian's with respect to their own strategies are positive definite. However, there are 3 isolated Nash equilibria.

Optimality conditions for individual optimization

Consider the ith player's individual optimization problem $\mathbf{F}_i(x_{-i})$ in (1.1), for given x_{-i}. For convenience, we write the constraining functions as

$$g_i(x_i) \; := \; (g_{i,1}(x_i), \ldots, g_{i,m_i}(x_i)).$$

Suppose $\mathbf{x}^* = (x_1^*, \ldots, x_N^*)$ is a NE. Under linear independence constraint qualification condition (LICQC) at x_i^*, there exist Lagrange multipliers $\lambda_{i,j}$ such that

$$\left\{ \begin{array}{c} \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x_i^*) = \nabla_{x_i} f_i(\mathbf{x}^*), \\ 0 \le \lambda_{i,j} \perp g_{i,j}(x_i^*) \ge 0 \, (j \in \mathcal{I}_i). \end{array} \right. \tag{2.1}$$

The above is the Karush-Kuhn-Tucker (KKT) condition for the optimization $\mathbf{F}_i(x_{-i}^*)$.

Optimality conditions for individual optimization

Therefore, $\mathbf{x}^*$ and $\lambda_{i,j}$ satisfy the following polynomial system for all $i = 1, \ldots, N$

$$\underbrace{\begin{bmatrix} \nabla_{x_i} g_{i,1} & \nabla_{x_i} g_{i,2} & \cdots & \nabla_{x_i} g_{i,m_i} \\ g_{i,1}(x) & 0 & \cdots & 0 \\ 0 & g_{i,2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{i,m_i}(x) \end{bmatrix}}_{G_i(x_i)} \underbrace{\begin{bmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,m_i} \end{bmatrix}}_{\lambda_i} = \underbrace{\begin{bmatrix} \nabla_{x_i} f_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\hat{f}_i(\mathbf{x})}. \tag{2.2}$$

If there exists a matrix polynomial $H_i(x_i)$ such that

$$H_i(x_i) G_i(x_i) = I_{m_i}, \tag{2.3}$$

then we can express λ_i as

$$\lambda_i = H_i(x_i) G_i(x_i) \lambda_i = H_i(x_i) \hat{f}_i(\mathbf{x}). \tag{2.4}$$

Optimality conditions for individual optimization

Interestingly, the matrix polynomial $H_i(x_i)$ satisfying (2.3) exists under the nonsingularity condition on g_i. The polynomial tuple g_i is said to be *nonsingular* if $G_i(x_i)$ has full column rank for all $x_i \in \mathbb{C}^{n_i}$ [1]. It is a generic condition. We remark that if g_i is nonsingular, then the LICQC holds at every minimizer of (1.1), so there must exist $\lambda_{i,j}$ satisfying (2.1).

Optimality conditions for individual optimization

Assume that every constraining polynomial tuple g_i is nonsingular. Then $\lambda_{i,j}(\mathbf{x})$ can be expressed as polynomials as in (2.4), and each Nash equilibrium $\mathbf{x}^*$ satisfies the following polynomial systems $(i = 1, \ldots, N)$

$$(S_i) : \begin{cases} \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{i,j}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0, \\ g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i), \; \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \, (j \in \mathcal{I}_i), \\ g_{i,j}(x_i) \ge 0 \, (j \in \mathcal{I}_i), \; \lambda_{i,j}(\mathbf{x}) \ge 0 \, (j \in \mathcal{I}_i). \end{cases} \tag{2.5}$$

The above are necessary conditions for NEs. When every optimization in (1.1) is convex, the (2.5) are sufficient conditions for NEs.

Optimization based on KKT conditions

Let $[\mathbf{x}]_d$ be the vector of monomials in $\mathbf{x}$ whose degree are not greater than d. Choose a generic positive definite matrix

$$\Theta \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Then we consider the following optimization problem

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{x}} & [\mathbf{x}]_1^T \cdot \Theta \cdot [\mathbf{x}]_1 \\ s.t. & \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{i,j}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0 \, (i \in [N]), \\ & g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]). \end{array}\right. \tag{2.6}$$

Optimization based on KKT conditions

Under the nonsingularity assumptions on g_i, every Nash equilibrium $\mathbf{x}^*$ is a feasible point of (2.6), while the converse is typically not true. However, for every feasible point $\mathbf{x}$ of (2.6), the x_i is a critical point for the optimization $\mathbf{F}_i(x_{-i})$. It is important to observe that if (2.6) is infeasible, then there are no NEs. If (2.6) is feasible, then it must have a minimizer, because its objective is a positive definite quadratic function. Moreover, for generic Θ, the minimizer of (2.6) is unique.

Checking the Nash equilibrium

Assume that $u := (u_1, \ldots, u_N)$ is an optimizer of (2.6). If each u_i is a minimizer for the optimization problem $\mathbf{F}_i(u_{-i})$, then u is a NE. To this end, for each player, consider the optimization problem:

$$\left\{ \begin{array}{rl} \omega_i := \min & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ s.t. & g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i), \\ & g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i). \end{array} \right. \tag{2.7}$$

If all the optimal values $\omega_i \geq 0$, then u is a Nash Equilibrium. However, if one of them is negative, say, $\omega_i < 0$, then u is not a NE. Let U_i be a set of some optimizers of (2.7), then u violates the following inequalities

$$f_i(x_i, x_{-i}) \leq f_i(v, x_{-i}) \quad (v \in U_i). \tag{2.8}$$

However, every Nash equilibrium must satisfy (2.8).

Excluding a point which is not a NE

When u is not a NE, we aim at finding a new candidate by posing the inequalities in (2.8). Therefore, we consider the following optimization problem:

$$\begin{cases} \min\limits_{\mathbf{x}} & [\mathbf{x}]_1^T \cdot \Theta \cdot [\mathbf{x}]_1 \\ s.t. & \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{ij}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0 \, (i \in [N]), \\ & g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & f_i(v, x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \; (v \in \mathcal{K}_i, i \in [N]). \end{cases} \tag{2.9}$$

In the above, each $\mathcal{K}_i$ is a set containing optimizers of (2.7). If the minimizer of (2.9) is verified to be a NE, then we are done. Otherwise, we can add more inequalities like (2.8).

Repeating this procedure, we get the following algorithm.

Algorithm 2.1 (Finding one Nash equilibrium)

For the NEP given as in (1.1), do the following

- S.0 Initialize $\mathcal{K}_i := \emptyset$ for all i and $l := 0$. Choose a generic positive definite matrix Θ of length $n+1$.
- S.1 Solve the polynomial optimization problem (2.9). If it is infeasible, there is no NE and stop; otherwise, solve it for an optimizer u.
- S.2 For each $i = 1, \ldots, N$, solve the optimization (2.7). If all $\omega_i \geq 0$, u is a NE and stop. If one of ω_i is negative, go to the next step.
- S.3 For each i with $\omega_i < 0$, obtain a set U_i of some (may not all) optimizers of (2.7); then update the set $\mathcal{K}_i := \mathcal{K}_i \cup U_i$. Let $l := l + 1$, then go to Step 1.

Finding one Nash equilibrium

Theorem 2.2 (J. Nie, X. Tang)

Assume each constraining polynomial tuple g_i is nonsingular and let $\lambda_{i,j}(\mathbf{x})$ be Lagrange multiplier polynomials as in (2.4). Let $\mathcal{G}$ be the feasible set of (2.6) and $\mathcal{G}^*$ be the set of all NEs. If the complement $\mathcal{G}\backslash\mathcal{G}^*$ is a finite set, i.e., the cardinality $\ell := |\mathcal{G}\backslash\mathcal{G}^*| < \infty$, then Algorithm 2.1 must terminate within at most ℓ loops.

Finding one Nash equilibrium

Corollary 2.3 (J. Nie, X. Tang)

Assume each g_i is a nonsingular tuple of polynomials. Suppose each $g_{i,j}(x_i)$ $(j \in \mathcal{E}_i)$ is linear, each $g_{i,j}(x_i)$ $(j \in \mathcal{I}_i)$ is concave, and each $f_i(x_i, x_{-i})$ is convex in x_i for given x_{-i}. Then Algorithm 2.1 must terminate at the first loop with $l = 0$, returning a NE or reporting that there is no NE.

More Nash Equilibria

Assume that $\mathbf{x}^*$ is a Nash Equilibrium produced by Algorithm 2.1, i.e., $\mathbf{x}^*$ is also a minimizer of (2.9). Note that all KKT points $\bar{\mathbf{x}}$ satisfying

$$[\bar{\mathbf{x}}]_1^T \Theta [\bar{\mathbf{x}}]_1 \ < \ [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1$$

are excluded from the feasible set of (2.9) by the constraints

$$f_i(u_i, x_{-i}) - f_i(x_i, x_{-i}) \ \geq \ 0 \quad (\forall\, u \in \mathcal{K}_i, \forall\, i \in [N]).$$

If $\mathbf{x}^*$ is an isolated NE (e.g., this is the case if there are finitely many NEs), there exists a scalar $\delta > 0$ such that

$$[\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \ \geq \ [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1 + \delta$$

for all other NEs $\mathbf{x}$.

More Nash Equilibria

For such δ, we can try to find a different NE by solving the following optimization problem

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{x}} & [\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \\ s.t. & \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{ij}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0 \, (i \in [N]), \\ & g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & \lambda_{i,j}(\mathbf{x}) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\ & f_i(v, x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \; (v \in \mathcal{K}_i, i \in [N]), \\ & [\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \; \geq \; [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1 + \delta. \end{array}\right. \tag{2.10}$$

When an optimizer of (2.10) is computed, we can check if it is a NE or not by solving (2.7). If it is, we get a new NE that is different from $\mathbf{x}^*$. Otherwise, we can union new points to $\mathcal{K}_i$.

More Nash Equilibria

A concern in computation is how to choose the constant $\delta > 0$ for (2.10). We want a value $\delta > 0$ such that there is no other Nash equilibrum u such that $[u]_1^T \Theta [u]_1 \leq [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1 + \delta$. To this end, we consider the following maximization problem

$$
\left\{\begin{array}{cl}
\max\limits_{\mathbf{x}} & [\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \\
s.t. & \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{ij}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0 \, (i \in [N]), \\
& g_{i,j}(x_i) = 0 \, (j \in \mathcal{E}_i, i \in [N]), \\
& \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \, (j \in \mathcal{I}_i, i \in [N]), \\
& g_{i,j}(x_i) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\
& \lambda_{i,j}(\mathbf{x}) \geq 0 \, (j \in \mathcal{I}_i, i \in [N]), \\
& f_i(v, x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \; (v \in \mathcal{K}_i, i \in [N]), \\
& [\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \leq [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1 + \delta.
\end{array}\right. \tag{2.11}
$$

More Nash Equilibria

Interestingly, if $\mathbf{x}^*$ is also a maximizer of (2.11), then the feasible set of (2.10) contains all NEs except $\mathbf{x}^*$, under some general assumptions.

Proposition 2.4 (J. Nie, X. Tang)

Assume Θ is a generic positive definite matrix and $\mathbf{x}^*$ is a minimizer of (2.9).

(i) If $\mathbf{x}^*$ is also a maximizer of (2.11), then there is no other Nash Equilibrium u satisfying $[u]_1^T \Theta [u]_1 \le [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1 + \delta$.

(ii) If $\mathbf{x}^*$ is an isolated KKT point, then there exists $\delta > 0$ such that $\mathbf{x}^*$ is also a maximizer of (2.11).

More Nash Equilibria

Algorithm 2.5 (Finding more Nash Equilibiria)

Give an initial value for δ (say, 0.1).

- S.1 Solve the maximization problem (2.11). If its optimal value η equals $\upsilon := [\mathbf{x}^*]_1^T \Theta [\mathbf{x}^*]_1$, then go to Step 2. If η is bigger than υ, then let $\delta = \min(\delta/5, \eta - \upsilon)$ and repeat this step.
- S.2 Solve the optimization problem (2.10). If it is infeasible, then there are no additional NEs; if it is feasible, solve it for a minimizer u.
- S.3 For each $i = 1, \ldots, N$, solve the optimization (2.7) for the optimal value ω_i. If all $\omega_i \geq 0$, stop and u is a NE. If one of ω_i is negative, go to Step 4.
- S.4 For each $i \in [N]$, update the set $\mathcal{K}_i := \mathcal{K}_i \cup U_i$, and then go back to Step 2.

More Nash Equilibria

Theorem 2.6 (J. Nie, X. Tang)

Under the same assumptions in Theorem 2.2, if Θ is a generic positive definite matrix and $\mathbf{x}^*$ is an isolated KKT point, then Algorithm 2.5 must terminate after finitely many steps, returning a NE that is different from $\mathbf{x}^*$ or reporting the nonexistence of other NEs.

More Nash Equilibria

Once a new NE is obtained, we can repeatedly apply Algorithm 2.5, to compute more NEs, if they exist. In particular, if there are finitely many NEs, we can eventually get all of them.

Corollary 2.7 (J. Nie, X. Tang)

Under the assumptions of Theorem 2.6, if there are finitely many Nash equilibria, then all of them can be found by applying Algorithm 2.5 repeatedly.

More Nash Equilibria

Theorem 2.8 (J. Nie, X. Tang)

Let $d_{i,j} > 0$, $a_{i,j} > 0$ be degrees, for $i \in [N], j \in [m_i]$. If each $g_{i,j}$ is a generic polynomial in x_i of degree $d_{i,j}$ and each f_i is a generic polynomial in $\mathbf{x}$ and its degree in x_j is $a_{i,j}$, then the KKT system has finitely many complex solutions and hence the NEP has finitely many KKT points.

Localizing and moment matrices

Let $\mathbb{R}^{\mathbb{N}^n_{2d}}$ denote the space of all real vectors that are labeled by $\alpha \in \mathbb{N}^n_{2d}$. Each $y \in \mathbb{R}^{\mathbb{N}^n_{2d}}$ is labeled as

$$y = (y_\alpha)_{\alpha \in \mathbb{N}^n_{2d}}.$$

Such y is called a *truncated multi-sequence* (tms) of degree $2d$. For a polynomial $f = \sum_{\alpha \in \mathbb{N}^n_{2d}} f_\alpha z^\alpha \in \mathbb{R}[z]_{2d}$, define the operation

$$\langle f, y \rangle = \sum_{\alpha \in \mathbb{N}^l_{2d}} f_\alpha y_\alpha. \tag{3.1}$$

The operation $\langle f, y \rangle$ is a bilinear function in (f, y).

Localizing and moment matrices

For a polynomial $q \in \mathbb{R}[z]$ with $\deg(q) \leq 2k$ and the integer

$$t \ = \ k - \lceil \deg(q)/2 \rceil,$$

the outer product $q \cdot [z]_t([z]_t)^T$ is a symmetric matrix polynomial in z, with length $\binom{n+t}{t}$. We write the expansion as

$$q \cdot [z]_t([z]_t)^T \ = \sum_{\alpha \in \mathbb{N}^n_{2d}} z^\alpha Q_\alpha,$$

for some symmetric matrices Q_α. Then we define the matrix function

$$L_q^{(d)}[y] \ := \sum_{\alpha \in \mathbb{N}^n_{2d}} y_\alpha Q_\alpha. \tag{3.2}$$

It is called the kth *localizing matrix* of q and generated by y. For given q, $L_q^{(d)}[y]$ is linear in y. Clearly, if $q(u) \geq 0$ and $y = [u]_{2d}$, then

$$L_q^{(d)}[y] = q(u)[u]_t[u]_t^T \succeq 0.$$

Localizing and moment matrices

When q is the constant one polynomial, the localizing matrix $L_1^{(d)}[y]$ reduces to a moment matrix, which we denote as

$$M_d[y] := L_1^{(d)}[y].$$

For instance, for $n = 2$ and $y \in \mathbb{R}^{\mathbb{N}_4^2}$, we have

$$M_2[y] = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}.$$

The optimization for all players

First, we discuss how to solve (2.9). Suppose the set $\mathcal{K}_i$ is given, for each player. For notational convenience, denote the polynomial tuples

$$
\begin{aligned}
\Phi_i &:= \left\{ \nabla_{x_i} f_i(\mathbf{x}) - \textstyle\sum_{j=1}^{m_i} \lambda_{ij}(\mathbf{x}) \nabla_{x_i} g_{ij} \right\} \\
&\qquad \cup \left\{ g_{i,j} : j \in \mathcal{E}_i \right\} \cup \left\{ \lambda_{i,j}(\mathbf{x}) \cdot g_{i,j} : j \in \mathcal{I}_i \right\}, \\
\Psi_i &:= \left\{ f_i(v, x_{-i}) - f_i(x_i, x_{-i}) : \ v \in \mathcal{K}_i \right\} \\
&\qquad \cup \left\{ g_{i,j} : j \in \mathcal{I}_i \right\} \cup \left\{ \lambda_{i,j}(\mathbf{x}) : j \in \mathcal{I}_i \right\}.
\end{aligned} \tag{3.3}
$$

And the unions

$$
\Phi := \bigcup_{i=1}^{N} \Phi_i, \quad \Psi := \bigcup_{i=1}^{N} \Psi_i. \tag{3.4}
$$

The optimization for all players

Then, the optimization (2.9) can be equivalently written as

$$\left\{\begin{array}{rl} \vartheta_{\min} := \min\limits_{\mathbf{x}\in\mathbb{R}^n} & \theta(\mathbf{x}) := [x]_1^T \Theta [x]_1 \\ s.t. & p(\mathbf{x}) = 0 \ (\forall\, p \in \Phi), \\ & q(\mathbf{x}) \geq 0 \ (\forall\, q \in \Psi). \end{array}\right. \tag{3.5}$$

The optimization for all players

Denote the degree

$$d_0 := \max\{\lceil \deg(p)/2 \rceil : p \in \Phi \cup \Psi\}.$$

For a degree $k \geq d_0$, consider the the kth order moment relaxation for (3.5)

$$\left\{ \begin{array}{rll} \vartheta_k := & \min\limits_{y} & \langle \theta, y \rangle \\ & s.t. & y_0 = 1,\ L_p^{(k)}[y] = 0\,(p \in \Phi), \\ & & M_d[y] \succeq 0,\ L_q^{(k)}[y] \succeq 0\,(q \in \Psi), \\ & & y \in \mathbb{R}^{\mathbb{N}^n_{2k}}. \end{array} \right. \tag{3.6}$$

The optimization for all players

Algorithm 3.1

Let θ, Φ, Ψ be as in (3.5). Initialize $k := d_0$.

Step 1 Solve the moment relaxation (3.6). If it is infeasible, (3.5) has no feasible points and stop; otherwise, solve it for a minimizer y^*.

Step 2 Let $u = (y^*_{e_1}, \ldots, y^*_{e_n})$. If u is feasible for (3.5) and $\vartheta_k = \theta(u)$, then u is a minimizer of (3.5). Otherwise, let $k := k + 1$ and go to Step 1.

The optimization for all players

Theorem 3.2 (J. Nie, X. Tang)

Assume the matrix Θ is a generic positive definite matrix and $\text{Ideal}[\Phi] + \text{Qmod}[\Psi]$ is archimedean.

(i) If the polynomial optimization (3.5) is infeasible, then the moment relaxation (3.6) must be infeasible when the order k is big enough.

(ii) Suppose the optimization (3.5) is feasible. Let $u^{(k)}$ be the point u produced in the Step 2 of Algorithm 3.1 in the kth loop. Then $u^{(k)}$ converges to the unique minimizer of (3.5). In particular, if the real zero set of Φ is finite, then $u^{(k)}$ is the unique minimizer of (3.5), when k is sufficiently large.

Numerical experiments

Example 4.1

$$\text{1st player:} \begin{cases} \min\limits_{x_1 \in \mathbb{R}^3} & \sum_{j=1}^3 x_{1,j}(x_{1,j} - j \cdot x_{2,j}) \\ s.t. & 1 - x_{1,1}x_{1,2} \geq 0,\ 1 - x_{1,2}x_{1,3} \geq 0,\ x_{1,1} \geq 0, \end{cases}$$

$$\text{2nd player:} \begin{cases} \min\limits_{x_2 \in \mathbb{R}^3} & \prod_{j=1}^3 x_{2,j} + \sum\limits_{\substack{1 \leq i < j \leq 3 \\ 1 \leq k \leq 3}} x_{1,i}x_{1,j}x_{2,k} + \sum\limits_{\substack{1 \leq i \leq 3 \\ 1 \leq j < k \leq 3}} x_{1,i}x_{2,j}x_{2,k} \\ s.t. & 1 - (x_{2,1})^2 - (x_{2,2})^2 = 0. \end{cases}$$

The first player's optimization is non-convex, with an unbounded feasible set.

The Lagrange multipliers can be expressed as

$$\lambda_{1,1} = (1 - x_{1,1}x_{1,2})\tfrac{\partial f_1}{\partial x_{1,1}}, \quad \lambda_{1,2} = -x_{1,1}\tfrac{\partial f_1}{\partial x_{1,2}},$$
$$\lambda_{1,3} = x_{1,1}\tfrac{\partial f_1}{\partial x_{1,1}} - x_{1,2}\tfrac{\partial f_1}{\partial x_{1,2}}, \quad \lambda_2 = -\tfrac{1}{2}(x_2^T \nabla_{x_2} f_2).$$

Applying Algorithm 2.5, we get four NEs:

$$\begin{array}{ll} x_1^* = (0.3198, 0.6396, -0.6396), & x_2^* = (0.6396, 0.6396, -0.4264); \\ x_1^* = (0.0000, 0.3895, 0.5842), & x_2^* = (-0.8346, 0.3895, 0.3895); \\ x_1^* = (0.2934, -0.5578, 0.8803), & x_2^* = (0.5869, -0.5578, 0.5869); \\ x_1^* = (0.0000, -0.5774, -0.8660), & x_2^* = -(0.5774, 0.5774, 0.5774). \end{array}$$

Their accuracy parameters (the biggest absolute value of the minimum of (2.7)) are respectively

$$7.1879 \cdot 10^{-8},\ 3.5040 \cdot 10^{-7},\ 4.3732 \cdot 10^{-7}, 6.4360 \cdot 10^{-7}.$$

It took about 30 seconds.

If the second player's objective becomes

$$-\prod_{j=1}^{3} x_{2,j} + \sum_{\substack{1\le i\le 3\\ 1\le j<k\le 3}} x_{1,i}x_{2,j}x_{2,k} - \sum_{\substack{1\le i<j\le 3\\ 1\le k\le 3}} x_{1,i}x_{1,j}x_{2,k},$$

then there is no NE, which is detected by Algorithm 2.1. It took around 16 seconds.

Example 4.2

$$\text{1st player:} \begin{cases} \min\limits_{x_1 \in \mathbb{R}^2} & (2x_{1,1} - x_{1,2} + 3)x_{1,1}x_{2,1} \\ & \qquad +[(2x_{1,2})^2 + (x_{3,2})^2]x_{1,2} \\ s.t. & 1 - x_1^T x_1 \geq 0, \end{cases}$$

$$\text{2nd player:} \begin{cases} \min\limits_{x_2 \in \mathbb{R}^2} & [(x_{2,1})^2 - x_{1,2}]x_{2,1} \\ & \qquad +[(x_{2,2})^2 + 2x_{3,2} + x_{1,2}x_{3,1}]x_{2,2} \\ s.t. & x_2^T x_2 - 1 = 0,\ x_{2,1} \geq 0,\ x_{2,2} \geq 0, \end{cases}$$

$$\text{3rd player:} \begin{cases} \min\limits_{x_3 \in \mathbb{R}^2} & (x_{1,1}x_{1,2} - 1)x_{3,1} - [3(x_{3,2})^2 + 1]x_{3,2} \\ & \qquad +2[x_{3,1} + x_{3,2}]x_{3,1}x_{3,2} \\ s.t. & 1 - (x_{3,1})^2 \geq 0,\ 1 - (x_{3,2})^2 \geq 0. \end{cases}$$

The Lagrange multipliers can be represented as

$$\begin{array}{lll} \lambda_{1,1} = -\frac{1}{2}(x_1^T \nabla_{x_1} f_1), & \lambda_{2,1} = \frac{1}{2}(x_2^T \nabla_{x_2} f_2), & \lambda_{2,2} = \frac{\partial f_2}{x_{2,1}} - 2x_{2,1}\lambda_{2,1}, \\ \lambda_{2,3} = \frac{\partial f_2}{x_{2,2}} - 2x_{2,2}\lambda_{2,1}, & \lambda_{3,1} = -\frac{x_{3,1}}{2}\frac{\partial f_3}{\partial x_{3,1}}, & \lambda_{3,2} = -\frac{x_{3,2}}{2}\frac{\partial f_3}{\partial x_{3,2}}. \end{array}$$

Applying Algorithm 2.5, we get the unique NE

$$\begin{array}{l} x_1^* = (-0.3558, -0.9346), \\ x_2^* = (1.0000, 0.0000), \\ x_3^* = (-0.3331, 1.0000). \end{array}$$

The accuracy parameter is $9.2310 \cdot 10^{-9}$. It took around 9 seconds.

Example 4.3

Consider the 2-player NEP

$$\text{1st player:} \begin{cases} \min\limits_{x_1 \in \mathbb{R}^2} & 2x_{1,1}x_{1,2} + 3x_{1,1}(x_{2,1})^2 + 3(x_{1,2})^2 x_{2,2} \\ s.t. & (x_{1,1})^2 + (x_{1,2})^2 - 1 \geq 0, \\ & 2 - (x_{1,1})^2 - (x_{1,2})^2 \geq 0 \end{cases}$$

$$\text{2nd player:} \begin{cases} \min\limits_{x_2 \in \mathbb{R}^2} & (x_{2,1})^3 + (x_{2,2})^3 + x_{1,1}(x_{2,1})^2 \\ & \qquad + x_{1,2}(x_{2,2})^2 + x_{1,1}x_{1,2}(x_{2,1} + x_{2,2}) \\ s.t. & (x_{2,1})^2 + (x_{2,2})^2 - 1 \geq 0, \\ & 2 - (x_{2,1})^2 + (x_{2,2})^2 \geq 0. \end{cases}$$

The Lagrange multipliers can be represented as $(i = 1, 2)$:

$$\lambda_{i,1} = \frac{1}{2}\nabla_{x_i} f_i^{\,T} x_i(2 - x_i^T x_i), \quad \lambda_{i,2} = \frac{1}{4}\nabla_{x_i} f_i^{\,T} x_i(1 - x_i^T x_i).$$

By Algorithm 2.5, we get the unique NE

$$x_1^* = (-1.3339, 0.4698), \quad x_2^* = (-1.4118, 0.0820),$$

with the accuracy parameter $3.5186 \cdot 10^{-8}$. It took around 5 seconds.

Example 4.4

Consider the unconstrained NEP

$$\text{1st player:}\begin{cases} \min & \sum_{i=1}^{n_1}(x_{1,i})^4 + \sum_{0\le i\le j\le k\le n_1} \frac{x_{1,i}x_{1,j}(x_{1,k}+x_{2,i}+x_{3,j})}{(n_1)^2} \\ s.t. & x_1 \in \mathbb{R}^{n_1}, \end{cases}$$

$$\text{2nd player:}\begin{cases} \min & \sum_{i=1}^{n_2}(x_{2,i})^4 + \sum_{0\le i\le j\le k\le n_2} \frac{x_{2,i}x_{2,j}(x_{2,k}+x_{3,i}+x_{1,j})}{(n_2)^2} \\ s.t. & x_2 \in \mathbb{R}^{n_2}, \end{cases}$$

$$\text{3rd player:}\begin{cases} \min & \sum_{i=1}^{n_3}(x_{3,i})^4 + \sum_{0\le i\le j\le k\le n_3} \frac{x_{3,i}x_{3,j}(x_{3,k}+x_{1,i}+x_{2,j})}{(n_3)^2} \\ s.t. & x_3 \in \mathbb{R}^{n_3}, \end{cases}$$

where $x_{1,0} = x_{2,0} = x_{3,0} = 1$, and $n_1 = n_2 = n_3$.

We implement Algorithm 2.5 for the cases $n_i = 2, \cdots, 6$. The computational results are shown in the following table. For all cases, we computed a unique NE successfully and obtained that $x_1^* = x_2^* = x_3^*$ (up to round-off errors).

n_1	$x_1^* = x_2^* = x_3^*$	ω^*	time
2	$(-0.8410, \quad -0.7125)$	$-8.8291 \cdot 10^{-9}$	0.34
3	$(-0.6743, -0.6157, -0.5236)$	$-6.6507 \cdot 10^{-9}$	1.58
4	$(-0.5950, -0.5606$ $-0.5097, -0.4363)$	$-1.0577 \cdot 10^{-9}$	16.86
5	$(-0.5476, -0.5247, -0.4919,$ $-0.4472, -0.3860)$	$-4.4438 \cdot 10^{-9}$	177.63
6	$(-0.5157, -0.4992, -0.4762,$ $-0.4457, -0.4060, -0.3534)$	$-3.7536 \cdot 10^{-9}$	1379.27

The time is displayed in seconds.

Example 4.5

Consider the NEP of the electricity market problem [3]. There are three generating companies, and the ith company possesses s_i generating units. For the ith company, the power generation of his jth generating unit is denoted by $x_{i,j}$, whose maximum capacity is $E_{i,j}$, and the cost is $\frac{1}{2}c_{i,j}(x_{i,j})^2 + d_{i,j}x_{i,j}$, where $E_{i,j}, c_{i,j}, d_{i,j}$ are parameters. The electricty price is given by

$$\phi(\mathbf{x}) \; := \; b - a(\sum_{i=1}^{3}\sum_{j=1}^{s_i} x_{i,j}).$$

The aim of the each company is to solve the following optimization problem:

$$i\text{th player:} \left\{ \begin{array}{cl} \min\limits_{x_i \in \mathbb{R}^{s_i}} & \frac{1}{2}\sum_{j=1}^{s_i}(c_{i,j}(x_{i,j})^2 + d_{i,j}x_{i,j}) - \phi(\mathbf{x})(\sum_{j=1}^{s_i} x_{i,j}). \\ s.t. & 0 \le x_{i,j} \le E_{i,j} \, (j \in [s_i]). \end{array} \right.$$

The Lagrange multipliers according to the constraints $g_{i,2j-1} := E_{i,j} - x_{i,j} \geq 0, g_{i,2j} := x_{i,j} \geq 0$ can be represented as

$$\lambda_{i,2j-1} = -\frac{\partial f_i}{\partial x_{i,j}} \cdot x_{i,j}/E_{i,j}, \lambda_{i,2j} = \frac{\partial f_i}{\partial x_{i,j}} + \lambda_{i,2j-1}. \, (j \in [s_i])$$

We run Algorithm 2.5 for the following setting:

$$\begin{array}{llllll}
s_i = i, & & a = 1, & & b = 10, & \\
c_{1,1} = 0.4, & c_{2,1} = 0.35, & c_{2,2} = 0.35, & c_{3,1} = 0.46, & c_{3,2} = 0.5, & c_{3,3} = 0.5, \\
d_{1,1} = 2, & d_{2,1} = 1.75, & d_{2,2} = 1, & d_{3,1} = 2.25, & d_{3,2} = 3, & d_{3,3} = 3, \\
E_{1,1} = 2, & E_{2,1} = 2.5, & E_{2,2} = 0.67, & E_{3,1} = 1.2, & E_{3,2} = 1.8, & E_{3,3} = 1.6.
\end{array}$$

We found the unique NE

$$x_1^* = 1.7184, \quad x_2^* = (1.8413, 0.6700), \quad x_3^* = (1.2000, 0.0823, 0.0823).$$

The accuracy parameter is $5.1183 \cdot 10^{-7}$. It took about 8 seconds.

Conclusions

This paper studies Nash equilibrium problems that are given by polynomial functions. Algorithms 2.1 and 2.5 are proposed for computing one or all NEs. The Lasserre type Moment-SOS hierarchy of semidefinite relaxations are used to solve the appearing polynomial optimization problems. Under generic assumptions, we can compute a Nash equilibrium if it exists, and detect the nonexistence if there is none. Moreover, we can get all Nash equilibria if there are finitely many ones of them.

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THANK YOU!