# Math 281C Homework 7 Solutions

1. We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at another nuisance parameter problem. Find the LRT of size $\alpha$ for testing

$$H_0 : \gamma = 1 \quad \text{versus} \quad H_1 : \gamma \neq 1$$

based on a sample $X_1, \dots, X_n$ from the Weibull$(\gamma, \beta)$ with pdf

$$f(x) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}, \quad x > 0, \quad \beta > 0.$$

**Solution:** The restricted MLE is $\widehat{\beta} = \overline{X}$. The unrestricted MLE does not have an analytical solution. The rejection region can be derived based on asymptotic $\chi_1^2$ distribution.

2. Consider a linear regression model $Y = \beta_0 + \beta_1 X + \sigma\varepsilon$, where $\beta_0$ is the intercept, $\beta_1$ is the slope coefficient, $\sigma > 0$ is the residual standard deviation, and $\varepsilon$ is the (unobservable) random error satisfying $\varepsilon | X \sim N(0, 1)$. Assume $\beta_0, \beta_1, \sigma^2$ are all unknown. Find the LRT of size $\alpha$ for testing

$$H_0 : \beta_1 = 0 \quad \text{versus} \quad H_1 : \beta_1 \neq 0$$

based on independent observations $(X_1, Y_1), \dots, (X_n, Y_n)$ from $(X, Y)$.

**Solution:** This is a classical regression problem. $\lambda(\boldsymbol{x}) = (1 - r^2)^{n/2}$, where $r^2$ is the $R^2$ of the regression model. When the regression only has a single covariate, $R^2$ is the square of correlation between $X$ and $Y$. Besides, $F = r^2(n-2)/(1-r^2) \sim F_{1,n-2}$, or equivalently, $T = r\sqrt{(n-2)/(1-r^2)} \sim t_{n-2}$. We reject $H_0$ if $F > F_{1,n-2}^{1-\alpha}$.

3. A random sample $X_1, \dots, X_n$ is drawn from a Pareto population with pdf

$$f(x) = \frac{\theta \nu^\theta}{x^{\theta+1}} \mathbb{1}(x \geq \nu),$$

where $\theta, \nu > 0$.

(a) Find the MLEs of $\theta$ and $\nu$.

**Solution:** $\widehat{\theta} = n/T$, where $T$ is defined in part (b). $\widehat{\nu} = X_{(1)}$.

(b) Show that the LRT of

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1$$

has critical region of the form $\{\mathbf{x} : T(\mathbf{x}) < C_1 \text{ or } T(\mathbf{x}) > C_2\}$, where $0 < C_1 < C_2$ and

$$T = \log\left\{\prod_{i=1}^n X_i / X_{(1)}\right\}.$$

**Solution:** $\lambda(\boldsymbol{x}) = (T/n)^n e^{n-T}$. The function is first increasing then decreasing in $T$, so $\lambda(\boldsymbol{x}) < C$ is equivalent to the claimed region.

(c) Show that, under $H_0$, $2T$ has a chi-squared distribution, and find the number of degrees of freedom. (Hint: Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/X_{(1)}$ conditional on $X_{(1)}$. Put these $n-1$ terms together, and notice that the distribution of $T$ given $X_{(1)}$ does not depend on $X_{(1)}$, and hence is also the unconditional distribution of $T$.)

**Solution:** First, notice that

$$f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots, x_n) = n! \prod_{i=1}^n f(x_i).$$

The following arguments are conditioned on $H_0 : \theta = 1$. Define random variables $(Y_1, Y_2, \ldots, Y_n) := (X_{(1)}, X_{(2)}/X_{(1)}, \ldots, X_{(n)}/X_{(1)})$. By a Jacobian transformation, we have

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = n! f(y_1) \prod_{i=2}^{n} f(y_1 y_i) y_1^{n-1} = \frac{n! \nu^n}{y_1^{n+1} \prod_{i=2}^{n} y_i^2}$$

In addition, the marginal distribution of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n\{1 - F(x)\}^{n-1} f(x) = \frac{n\nu^n}{x^{n+1}}.$$

Combining the above two displays gives us that

$$f_{Y_2,\ldots,Y_n|Y_1=y_1}(y_2, \ldots, y_n) = \frac{(n-1)!}{\prod_{i=2}^{n} y_i^2},$$

which does not depend on $y_1$. This means $(Y_2, \ldots, Y_n)$ is jointly independent with $Y_1$. Moreover, it shows that $(Y_2, \ldots, Y_n) \sim (Z_{(1)}, \ldots, Z_{(n-1)})$, where $Z_1, \ldots, Z_{n-1}$ are i.i.d. following Pareto$(1, 1)$. We are ready to derive the distribution of $2T$.

$$2T = 2 \log \left\{ \prod_{i=2}^{n} X_{(i)}/X_{(1)} \right\} \sim 2 \log \left\{ \prod_{i=1}^{n-1} Z_{(i)} \right\} = 2 \log \left\{ \prod_{i=1}^{n-1} Z_i \right\} = 2 \sum_{i=1}^{n-1} \log Z_i.$$

Finally, it can be shown using a density transformation that $2 \log Z_i \sim \chi_2^2$, so $2T \sim \chi_{2n-2}^2$.