Math 281C Homework 5

Due: 5:00pm, May 6th

1. Given independent random variables $X_1, \dots, X_n$, define

$$W = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} = \frac{n^{1/2}\bar{X}}{\sqrt{\sum_{i=1}^n X_i^2/n}}, \quad T = \frac{n^{1/2}\bar{X}}{S},$$

where $S^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$. Show that the following identity holds

$$T = \left(\frac{n-1}{n}\right)^{1/2} \frac{W}{\sqrt{1 - W^2/n}}.$$

And W and T have a one-to-one correspondence.

2. Let $U_1/\sigma_1^2 \sim \chi^2_{d_1}$, and $U_2/\sigma_2^2 \sim \chi^2_{d_2}$, and they are independent. Suppose $\sigma_2^2/\sigma_1^2 = a$. Show that U_2/U_1 and $aU_1 + U_2$ are independent. In particular, if $\sigma_1 = \sigma_2$, U_2/U_1 and $U_1 + U_2$ are independent.

3. Suppose that random vector (X, Y) has probability density function

$$\frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} \mathbb{1}(xy > 0), \quad x, y \in \mathbb{R}.$$

Does (X, Y) possess a multivariate normal distribution? Find the marginal distributions.

4. Suppose that $X_m \sim \text{Binomial}(m, p_1), Y_n \sim \text{Binomial}(n, p_2)$ and they are independent. To test $H_0 : p_1 = p_2 = p$ for some predetermined $p \in (0, 1)$, consider the test statistic

$$C^2_{m,n} = \frac{(X_m - mp)^2}{mp(1-p)} + \frac{(Y_n - np)^2}{np(1-p)}.$$

   (a) Find the limit distribution of $C^2_{m,n}$ as $m, n \to \infty$;

   (b) How would you modify the test statistic if p were unknown? What is the limit distribution after modification?