Math 281C Homework 4 Solutions

1. Let $X_1, \ldots, X_n$ be i.i.d. from the Gamma distribution $\Gamma(\alpha, \gamma)$ with unknown α and γ, whose p.d.f. is

$$f(x) = \frac{1}{\Gamma(\alpha)\gamma^\alpha} x^{\alpha-1} e^{-x/\gamma} \mathbb{1}(x > 0).$$

   (i) Show that $\Gamma(\alpha, \gamma)$ belongs to an exponential family.

   Solution: It can be shown that

$$f(x) = \exp\{\alpha \ln x - \gamma^{-1} x - \alpha \ln \gamma - \ln \Gamma(\alpha)\} \cdot x^{-1} \mathbb{1}(x > 0).$$

   So it belongs to an exponential family with parameters $\theta = (\alpha, \gamma^{-1})$ and $T(x) = (\ln x, x)$.

   (ii) Find a sufficient statistic for (α, γ).

   Solution: By factorization theorem, a sufficient statistic is $T(\boldsymbol{x}) = (\sum_{i=1}^n \ln x_i, \sum_{i=1}^n x_i)$. Alternatively, $T(\boldsymbol{x}) = (\Pi_{i=1}^n x_i, \sum_{i=1}^n x_i)$ is also correct.

2. Let $X_1, \ldots, X_n$ be i.i.d. from $\Gamma(\alpha, \gamma)$.

   (i) For testing $H_0 : \alpha \le \alpha_0$ versus $H_1 : \alpha > \alpha_0$, and $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha \ne \alpha_0$, show that there exist UMP unbiased tests whose rejections are based on $W = \prod_{i=1}^n (X_i/\bar{X})$.

   Solution: Denote $U := \Pi_{i=1}^n X_i$ and $T := \sum_{i=1}^n X_i$, then $W = h(U, T) := U/(T/n)^n$. We will employ Theorem 6.2.1 in Lecture 10 to prove the claim. To this end, we first show that when $\alpha = \alpha_0$, W is independent of T.

   When $\alpha = \alpha_0$, since the exponential family is of full rank, it can be shown that $T = \sum_{i=1}^n X_i$ is sufficient and complete, and hence, boundedly complete. We then show W is ancillary. Using the density transformation rule, $X_i \sim \Gamma(\alpha_0, \gamma)$ implies $Z_i := X_i/\gamma \sim \Gamma(\alpha_0, 1)$, so

$$W = \prod_{i=1}^n (X_i/\bar{X}) = \prod_{i=1}^n \{(X_i/\gamma)/(\bar{X}/\gamma)\} = \prod_{i=1}^n (Z_i/\bar{Z}),$$

   where each Z_i is independent of γ. Consequently, W is independent of γ. The desired independence follows from Basu's Theorem (Theorem 6.1.1 in Lecture 10).

   We are now ready to apply Theorem 6.2.1. For the first test, notice that $h(u, t)$ is increasing in u for each t, so a UMPU test of size α takes the form

$$\phi(w) = \begin{cases} 1 & \text{when } w \ge c, \\ 0 & \text{when } w < c, \end{cases}$$

   where c satisfies $\mathbb{E}_{\alpha_0} \phi(W) = \alpha$.

   For the second test, notice that $h(u, t)$ is linear in u for each t, so a UMPU test of size α takes the form

$$\phi(w) = \begin{cases} 1 & \text{when } w \le c_1 \text{ or } w \ge c_2, \\ 0 & \text{when } c_1 < w < c_2, \end{cases}$$

   where c_1, c_2 satisfy $\mathbb{E}_{\alpha_0} \phi(W) = \alpha$ and $\mathbb{E}_{\alpha_0}[W\phi(W)] = \alpha \mathbb{E}_{\alpha_0} W$.

   (ii) For testing $H_0 : \gamma \le \gamma_0$ versus $H_1 : \gamma > \gamma_0$, show that a UMP unbiased test rejects H_0 when $\sum_{i=1}^n X_i > C(\prod_{i=1}^n X_i)$. Here, $C(t)$ is a function of t.

   Solution: This is a direct application of Theorem 5.3.3 in Lectures 7 and 8. A UMPU test of size α is

$$\phi(u, t) = \begin{cases} 1 & \text{when } u \ge c(t), \\ 0 & \text{when } u < c(t), \end{cases}$$

   where $c(t)$ satisfy $\mathbb{E}_{\gamma_0}[\phi(U, T) | T = t] = \alpha$ for any t.

3. Let X and Y be independently distributed according to negative binomial distributions $Nb(p_1, m)$ and $Nb(p_2, n)$ respectively, and let $q_i = 1 - p_i$.

(i) There exists a UMP unbiased test for testing $H_0 : p_1 \le p_2$ versus $H_0 : p_1 > p_2$.

Solution: Here m and n are fixed integers. The joint density is

$$\begin{aligned} f(x,y) &= \binom{x+m-1}{x}\binom{y+n-1}{y}\exp\{x\ln p_1 + y\ln p_2 + m\ln q_1 + n\ln q_2\} \\ &= \binom{x+m-1}{x}\binom{y+n-1}{y}\exp\{x\ln(p_1/p_2) + (x+y)\ln p_2 + m\ln q_1 + n\ln q_2\}. \end{aligned}$$

Denote $U = X$, $T = X + Y$, and $\theta = \log(p_1/p_2)$, then, the original test is equivalent to

$$H_0 : \theta \le 0 \quad \text{versus} \quad H_1 : \theta > 0.$$

By Theorem 5.3.3 (1), A UMPU test of size α is

$$\phi(u,t) = \begin{cases} 1 & \text{when } u > c(t), \\ \gamma(t) & \text{when } u = c(t), \\ 0 & \text{otherwise,} \end{cases}$$

where $c(t), \gamma(t)$ satisfy $\mathbb{E}_{\theta=0}[\phi(U,T)|T = t] = \alpha$.

(ii) Determine the conditional distribution required for testing H_0 when $m = n = 1$.

Solution: The conditional distribution required is the density of U given $T = t$ and $p_1 = p_2$. When $m = n = 1$, X and Y degenerate to geometric distribution, and under $p_1 = p_2$, $X + Y \sim \text{Nb}(p, 2)$, so we have

$$\begin{aligned} \mathbb{P}(U = u|T = t) &= \frac{\mathbb{P}(U = u, T = t)}{\mathbb{P}(T = t)} = \frac{\mathbb{P}(X = u, Y = t - u)}{\mathbb{P}(X + Y = t)} \\ &= \frac{\mathbb{P}(X = u)\mathbb{P}(Y = t - u)}{\mathbb{P}(X + Y = t)} = \frac{(1-p)p^u \cdot (1-p)p^{(t-u)}}{(t+1)(1-p)^2 p^t} = \frac{1}{t+1}. \end{aligned}$$