1. Let $X_1, \ldots, X_n$ be i.i.d. from the Gamma distribution $\Gamma(\alpha, \gamma)$ with unknown $\alpha$ and $\gamma$, whose p.d.f. is
\[
f(x) = \frac{1}{\Gamma(\alpha)\gamma^n} x^{\alpha-1} e^{-x/\gamma} 1(x > 0).
\]

(i) Show that $\Gamma(\alpha, \gamma)$ belongs to an exponential family.

**Solution:** It can be shown that
\[
f(x) = \exp \{ \alpha \ln x - \gamma^{-1} x - \alpha \ln \gamma - \ln \Gamma(\alpha) \} \cdot x^{-1} 1(x > 0).
\]

So it belongs to an exponential family with parameters $\theta = (\alpha^{-1}, \gamma)$ and $T(x) = (\ln x, x)$.

(ii) Find a sufficient statistic for $\Gamma(\alpha, \gamma)$.

**Solution:** By factorization theorem, a sufficient statistic is $T(x) = (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i)$. Alternatively, $T(x) = (\Pi_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i)$ is also correct.

2. Let $X_1, \ldots, X_n$ be i.i.d. from $\Gamma(\alpha, \gamma)$.

(i) For testing $H_0 : \alpha \leq \alpha_0$ versus $H_1 : \alpha > \alpha_0$, and $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha \neq \alpha_0$, show that there exist UMP unbiased tests whose rejections are based on $W = \Pi_{i=1}^{n}(X_i/\bar{X})$.

**Solution:** Denote $U := \Pi_{i=1}^{n}X_i$ and $T := \sum_{i=1}^{n}X_i$, then $W = h(U, T) := U/(T/n)^n$. We will employ Theorem 6.2.1 in Lecture 10 to prove the claim. To this end, we first show that when $\alpha = \alpha_0$, $W$ is independent of $T$.

When $\alpha = \alpha_0$, since the exponential family is of full rank, it can be shown that $T = \sum_{i=1}^{n}X_i$ is sufficient and complete, and hence, boundedly complete. We then show $W$ is ancillary. Using the density transformation rule, $X_i \sim \Gamma(\alpha_0, \gamma)$ implies $Z_i := X_i/\gamma \sim \Gamma(\alpha_0, 1)$, so
\[
W = \Pi_{i=1}^{n}(X_i/\bar{X}) = \Pi_{i=1}^{n}((X_i/\gamma)/((\bar{X}/\gamma)) = \Pi_{i=1}^{n}(Z_i/\bar{Z}),
\]
where each $Z_i$ is independent of $\gamma$. Consequently, $W$ is independent of $\gamma$. The desired independence follows from Basu’s Theorem (Theorem 6.1.1 in Lecture 10).

We are now ready to apply Theorem 6.2.1. For the first test, notice that $h(u, t)$ is increasing in $u$ for each $t$, so a UMPU test of size $\alpha$ takes the form
\[
\phi(w) = \begin{cases} 
1 & \text{when } w \geq c, \\
0 & \text{when } w < c,
\end{cases}
\]
where $c$ satisfies $\mathbb{E}_{\alpha_0} \phi(W) = \alpha$.

For the second test, notice that $h(u, t)$ is linear in $u$ for each $t$, so a UMPU test of size $\alpha$ takes the form
\[
\phi(w) = \begin{cases} 
1 & \text{when } w \leq c_1 \text{ or } w \geq c_2, \\
0 & \text{when } c_1 < w < c_2,
\end{cases}
\]
where $c_1, c_2$ satisfy $\mathbb{E}_{\alpha_0} \phi(W) = \alpha$ and $\mathbb{E}_{\alpha_0}[W \phi(W)] = \alpha \mathbb{E}_{\alpha_0} W$.

(ii) For testing $H_0 : \gamma \leq \gamma_0$ versus $H_1 : \gamma > \gamma_0$, show that a UMP unbiased test rejects $H_0$ when $\sum_{i=1}^{n}X_i > C(\Pi_{i=1}^{n}X_i)$. Here, $C(t)$ is a function of $t$.

**Solution:** This is a direct application of Theorem 5.3.3 in Lectures 7 and 8. A UMPU test of size $\alpha$ is
\[
\phi(u, t) = \begin{cases} 
1 & \text{when } u \geq c(t), \\
0 & \text{when } u < c(t),
\end{cases}
\]
where $c(t)$ satisfy $\mathbb{E}_{\gamma_0}[\phi(U, T)|T = t] = \alpha$ for any $t$. 

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**Math 281C Homework 4 Solutions**
3. Let $X$ and $Y$ be independently distributed according to negative binomial distributions $\text{Nb}(p_1, m)$ and $\text{Nb}(p_2, n)$ respectively, and let $q_i = 1 - p_i$.

(i) There exists a UMP unbiased test for testing $H_0 : p_1 \leq p_2$ versus $H_0 : p_1 > p_2$.

**Solution:** Here $m$ and $n$ are fixed integers. The joint density is

$$
    f(x, y) = \binom{x + m - 1}{x} \binom{y + n - 1}{y} \exp\{x \ln p_1 + y \ln p_2 + m \ln q_1 + n \ln q_2\} \\
    = \binom{x + m - 1}{x} \binom{y + n - 1}{y} \exp\{x \ln (p_1/p_2) + (x + y) \ln p_2 + m \ln q_1 + n \ln q_2\}.
$$

Denote $U = X$, $T = X + Y$, and $\theta = \log(p_1/p_2)$, then, the original test is equivalent to

$$
    H_0 : \theta \leq 0 \text{ versus } H_1 : \theta > 0.
$$

By Theorem 5.3.3 (1), A UMPU test of size $\alpha$ is

$$
    \phi(u, t) = \begin{cases} 
    1 & \text{when } u > c(t), \\
    \gamma(t) & \text{when } u = c(t), \\
    0 & \text{otherwise},
    \end{cases}
$$

where $c(t), \gamma(t)$ satisfy $E_{\theta=0}[\phi(U, T)|T = t] = \alpha$.

(ii) Determine the conditional distribution required for testing $H_0$ when $m = n = 1$.

**Solution:** The conditional distribution required is the density of $U$ given $T = t$ and $p_1 = p_2$. When $m = n = 1$, $X$ and $Y$ degenerate to geometric distribution, and under $p_1 = p_2$, $X + Y \sim \text{Nb}(p, 2)$, so we have

$$
    \mathbb{P}(U = u|T = t) = \frac{\mathbb{P}(U = u, T = t)}{\mathbb{P}(T = t)} = \frac{\mathbb{P}(X = u, Y = t - u)}{\mathbb{P}(X + Y = t)} \\
    = \frac{\mathbb{P}(X = u)\mathbb{P}(Y = t - u)}{\mathbb{P}(X + Y = t)} = \frac{(1 - p)p^u \cdot (1 - p)p^{t-u}}{(t + 1)(1 - p)^2p^t} = \frac{1}{t + 1}.
$$